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Class 40: Interpolation Formulas

Paper by Vaaler (?)

News: 20., 22., 27. April: physical Lectures + zoom

→ Problem set 6. (Interp. formula stuff.)

Paley-Wiener Theorem:

Let $f \in M(\mathbb{R})$. Then, are equivalent:

1) f has an extension as an entire function s.t.

$$|f(z)| \leq C e^{2\pi M|z|} \quad \forall z \in \mathbb{C}, \text{ some } C>0,\ M>0.$$

2) $\hat{f}(t) = 0$, $|t| \geq M$. ($\operatorname{Supp} \hat{f} \subset [-M, M]$.)

Lemma:

Let f be an entire function of exponential π, and $f \in M(\mathbb{R})$. Then:

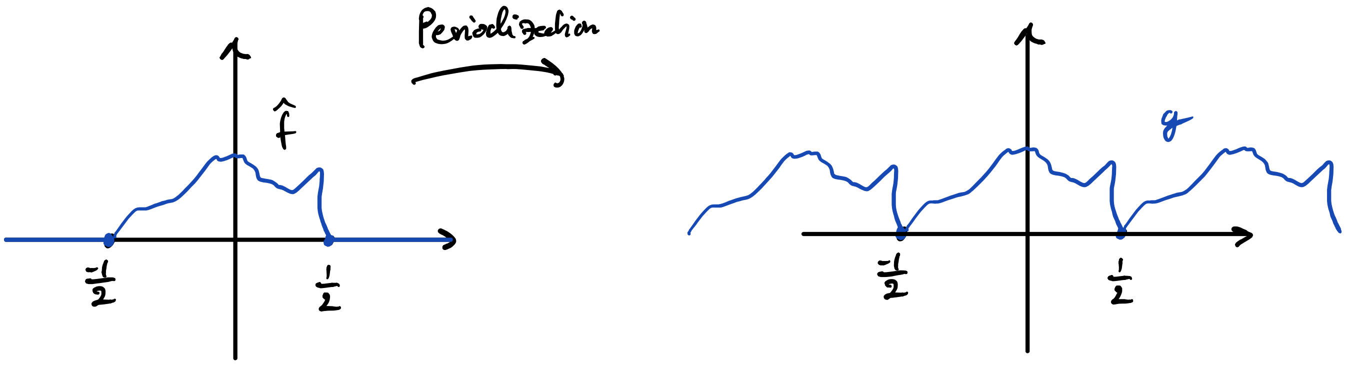
$$\sum_{n \in \mathbb{Z}} |f(n)|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

Corollary: Let f be an entire function of exp type π and $f \in M(\mathbb{R})$. Then:

if $f(n) = 0 \ \forall n \in \mathbb{Z} \Rightarrow f \equiv 0$.

Proof of Lemma:

From PWT, we know that supp $\hat{f} \subset [\frac{-1}{2}, \frac{1}{2}]$.



$$g(x) = \sum_{n \in \mathbb{Z}} \hat{f}(x+n)$$

g is a 1-periodic function.

g is a continuous function on $[\frac{-1}{2}, \frac{1}{2}]$.

Compute using same idea as PSF

$$\hat{g}(k) = \widehat{(\hat{f})}(k) = \int_{-\infty}^{\infty} \hat{f}(x) e^{-2\pi i k x} dx$$

PSF stuff:

$g(x) = \sum f(x+n)$

$\hat{g}(k) = \hat{f}(k)$

$$= \int_{-1/2}^{1/2} \hat{f}(x) e^{-2\pi i k x} dx$$

$$f, \hat{f} \in M(\mathbb{R}) \Rightarrow f(t) = \int_{-\infty}^{\infty} \hat{f}(x) e^{2\pi i t x} dx \quad \forall t \in \mathbb{R}$$

Fourier Inv. formula

So: $\hat{g}(k) = f(-k)$

$$\sum_{k\in\mathbb{Z}} |\hat{g}(k)| = \sum_{k\in\mathbb{Z}} |f(-k)| = \sum_{k\in\mathbb{Z}} |f(k)| \underset{f\in M(\mathbb{R})}{\leq} \sum_{k\neq 0} \frac{A}{|k|^{1+\alpha}} + |f(0)| < \infty.$$

Theorem 1 of Fourier series:

$$\Rightarrow \quad \sum_{k\in\mathbb{Z}} \hat{g}(k) e^{2\pi i k x} = g(x) \quad \forall x \in \left[\tfrac{-1}{2}, \tfrac{1}{2}\right].$$

Parseval Identity:

$$\sum_{n\in\mathbb{Z}} |\hat{g}(n)|^2 = \int_{-1/2}^{1/2} |g(x)|^2 dx$$

$$\|$$

$$\sum_{n\in\mathbb{Z}} |f(n)|^2 = \int_{-1/2}^{1/2} |\hat{f}(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \overset{\text{Plancherel}}{=} \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Proof of this Lemma is old exam question.

THEOREM: Let f be an entire function of exponential type π and $f \in M(\mathbb{R})$.

Then:

$$f(z) = \sum_{n\in\mathbb{Z}} \frac{\sin(\pi(z-n))}{\pi(z-n)} f(n).$$

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$$*) \quad f(z) = \frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n f(n)}{z-n}.$$

Proof: From proof of Lemma:

$$\begin{cases} g(x) = \sum_{n\in\mathbb{Z}} \hat{f}(x+n) \\ g \text{ is a 1-periodic function.} \\ g \text{ is a continuous function on } [\frac{-1}{2}, \frac{1}{2}] \\ \hat{g}(k) = f(-k) \end{cases}$$

$$\sum |\hat{g}(k)| < \infty$$

so: $\sum \hat{g}(k) e^{2\pi i k x} = g(x)$ uniformly convergence.

$$\sum_{k\in\mathbb{Z}} f(-k) e^{2\pi i k x} = \hat{f}(x)$$

$$\Rightarrow \sum_{k\in\mathbb{Z}} f(k) e^{-2\pi i k x} = \hat{f}(x), \quad x\in[\tfrac{-1}{2}, \tfrac{1}{2}]$$

F.I.F: $$f(x) = \int_{\mathbb{R}} \hat{f}(t) e^{2\pi i x t} dt = \int_{-1/2}^{1/2} \hat{f}(t) e^{2\pi i x t} dt$$

$$= \int_{-1/2}^{1/2} \Big(\sum_{k\in\mathbb{Z}} f(k) e^{-2\pi i k t} \Big) e^{2\pi i x t} dt$$

by uniform convergence.

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) \int_{-1/2}^{1/2} e^{2\pi i t(x-k)} dt$$

$$= \sum_{k \in \mathbb{Z}} f(k) \left\{ \frac{e^{2\pi i t(x-k)}}{2\pi i(x-k)} \Big|_{t=-1/2}^{1/2} \right\}$$

$$= \sum_{k \in \mathbb{Z}} f(k) \left\{ \frac{e^{\pi i(x-k)} - e^{-\pi i(x-k)}}{2\pi i\,(x-k)} \right\}$$

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin(\pi(x-k))}{\pi(x-k)} \quad \forall x \in \mathbb{R} \setminus \mathbb{Z}$$

BREAK

Now, define: $g_N(z) = \sum_{|k| \leq N} f(k) \frac{\sin(\pi(z-k))}{\pi(z-k)}$

g_N is entire and $g_N \xrightarrow[\text{in compacts}]{\text{uniformly}} \sum_{k \in \mathbb{Z}} f(k) \frac{\sin(\pi(z-k))}{\pi(z-k)}$

$g(z) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin(\pi(z-k))}{\pi(z-k)}$ is an entire function.

$\Rightarrow f(x) = g(x) \;\forall x \in \mathbb{R} \setminus \mathbb{Z} \Rightarrow$ by identity theorem: $f(z) = g(z)$.
from Complex Anal.

Lemma: Let $f \in C^2(\mathbb{R})$ s.t. $|f(x)| \le \frac{A}{|x|^{2\theta}}$, $|f''(x)| \le B$. $f:\mathbb{R}\to\mathbb{C}$
$x \ne 0$ $\forall x \in \mathbb{R}$

$\Rightarrow |f'(x)| \le \frac{C}{|x|^\theta}$ for $A>0, \theta>0$; for some $B>0 \Rightarrow$ for some $C>0$.
$x \ne 0$

Proof: Let $0 < h \le 1$, $x \in \mathbb{R}$ (and $|x| \ge 2$). Assume $f:\mathbb{R}\to\mathbb{R}$. (Ref, Im f in general case.)

MVT: $$f'(x) - f'(x-h) = f''(\theta)\cdot h \qquad \theta \in [x-h, x].$$

$$f'(x) - f'(x-h) \le B\cdot h$$

Let v s.t. $0 < v \le 1 \le \frac{|x|}{2} \Rightarrow$ $$\int_0^v f'(x)\,dh - \int_0^v f'(x-h)\,dh \le B\int_0^v h\,dh$$

$$v f'(x) - \int_{x-v}^{x} f'(y)\,dy \le B\left[\frac{h^2}{2}\right]_{h=0}^{v}$$

$$v f'(x) - \{f(x) - f(x-v)\} \le \frac{Bv^2}{2}$$

$$v f'(x) \le \frac{Bv^2}{2} + \frac{A}{|x|^{2\theta}} + \frac{A}{|x-v|^{2\theta}}$$ ← Using bounds of f.

Δ ineq

$$|x-v| \ge \big||x| - |v|\big| \ge |x| - |v| \ge |x| - \frac{|x|}{2} = \frac{|x|}{2}$$

$$v f'(x) \le \frac{Bv^2}{2} + \frac{A}{|x|^{2\theta}} + \frac{A\,2^\theta}{|x|^{2\theta}}$$

$$\Downarrow$$

$$v f'(x) \le \frac{Bv^2}{2} + \frac{\tilde{A}}{|x|^{2\theta}}$$

$$f'(x) \le \underbrace{\frac{Bv}{2} + \frac{\tilde{A}}{v|x|^{2\theta}}}.$$

Which v minimizes this quantity?

Choose: $v = \frac{1}{|x|^\theta} \Rightarrow$

$$f'(x) \leq \frac{B}{2|x|^{\theta}} + \frac{\tilde{A}}{|x|^{\theta}} = \frac{C}{|x|^{\theta}}.$$

THEOREM: (VAALER): f is an entire function of exp. type 2π.

And $f \in M(\mathbb{R})$.

$$\Rightarrow f(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n \in \mathbb{Z}} \frac{f(n)}{(z-n)^2} + \sum_{n \in \mathbb{Z}} \frac{f'(n)}{z-n} \right\}$$