(22.04.21)

Class 41: Vaaler's Formula and Harmonic Analysis.

Theorem: Let f be an entire function of exponential type 2π, s.t. $f \in M(\mathbb{R})$. [$\Rightarrow$ PWT: $\operatorname{supp} \hat{f} \subset [-1,1]$.]

Then: $$f(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m\in\mathbb{Z}} \frac{f(m)}{(z-m)^2} + \sum_{n\in\mathbb{Z}} \frac{f'(n)}{z-n} \right\}.$$

Vaaler's formula is the last topic on the exam.

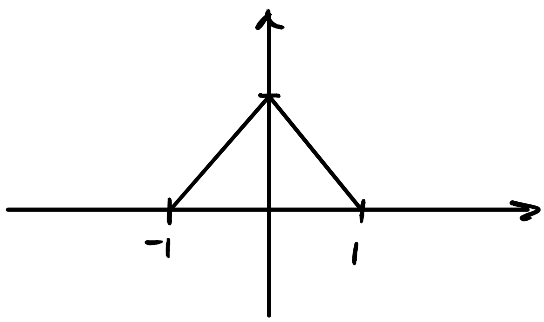
Proof: $K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2$ is of exponential type 2π:

$$|\sin \pi z|^2 \le \left|\frac{e^{i\pi z} - e^{-i\pi z}}{2i}\right|^2.$$

$$|z| \geq 1 \Rightarrow \left|\frac{\sin \pi z}{\pi z}\right|^2 \le \frac{(e^{\pi|z|})^2}{\pi^2} = \frac{e^{2\pi|z|}}{\pi^2}.$$

and $K(x) \subset M(\mathbb{R})$. $\overset{\text{PWT}}{\Rightarrow}$ $\hat{K}(\xi)$ has compact support.

$\hat{K}(t) = \max\{1-|t|, 0\}$:



Also $\hat{K} \subset M(\mathbb{R})$, so Fourier I.F.:

$$K(x) = \int_{-\infty}^{\infty} \hat{K}(t) e^{2\pi i t x}\, dt = \int_{-1}^{1} \hat{K}(t) e^{2\pi i t x}\, dt \quad \forall x \in \mathbb{R}.$$

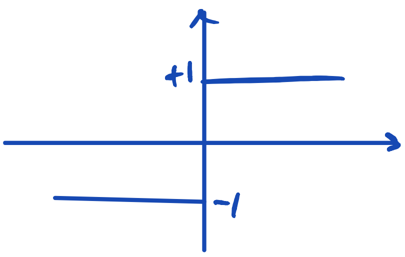
Let now $x \mapsto z$:

$$\underbrace{K(z)}_{\text{Entire func}} = \int_{-1}^{1} \underbrace{\hat{K}(t) e^{2\pi i t z}}_{\text{Entire func}}\, dt \quad \forall z \in \mathbb{C}.$$

$$(*) \qquad \left(\frac{\sin \pi z}{\pi z}\right)^2 = \int_{-1}^{1} (1-|t|) e^{2\pi i t z}\, dt \quad \forall z \in \mathbb{C}$$

$$(*) \qquad z\left(\frac{\sin \pi z}{\pi z}\right)^2 = \frac{1}{2\pi i} \int_{-1}^{1} \operatorname{sgn}(t) e^{2\pi i z t}\, dt$$

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$$



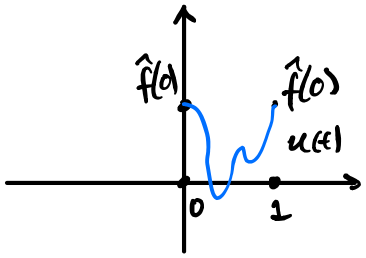
Let $t \in [0,1]$:

$$u(t) := \hat{f}(t) + \hat{f}(t-1). \xrightarrow{\text{Periodization}} \mathcal{U}(t) := \hat{f}(t) + \hat{f}(t-1)$$

$$v(t) := 2\pi i \{ t \hat{f}(t) + (t-1)\hat{f}(t-1) \}. \xrightarrow{\text{Periodization}} \mathcal{V}(t) = \ldots$$

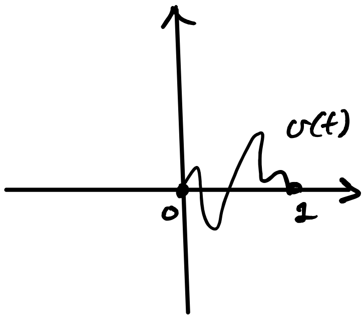
$$u(0) = \hat{f}(0) + \underbrace{\hat{f}(-1)}_{=0 \text{ by PWT.}} = \hat{f}(0) + 0 = \hat{f}(0).$$

$$u(1) = \hat{f}(1) + \hat{f}(0) = 0 + \hat{f}(0) = \hat{f}(0).$$



Also: $v(0) = 0$

$v(1) = 0$



So: u and v are 1-periodic functions & continuous.

Let $k \in \mathbb{Z}$:
$$\hat{u}(k) = \int_0^1 u(t) e^{-2\pi i tk} dt = \int_0^1 \hat{f}(t) e^{-2\pi i k t} dt + \int_0^1 \hat{f}(t-1) e^{-2\pi i k t} dt$$

$t = 1 - \omega$:

$$= \text{—— " ——} + \int_{-1}^{0} \hat{f}(\omega) e^{-2\pi i k(\omega+1)} d\omega$$

$$= \int_0^1 \hat{f}(t) e^{-2\pi i k t} dt + \int_{-1}^{0} \hat{f}(\omega) e^{-2\pi i \omega} d\omega$$

$$= \int_{-1}^{1} \hat{f}(t) e^{-2\pi i k t} dt$$

$$= \int_{-1}^{1} \hat{f}(t) e^{2\pi i t(-k)} dt$$

$f \in M(\mathbb{R})$, $\hat{f} \in M(\mathbb{R})$ $\quad = f(-k).$

Fourier Inversion Formula

$$\hat{v}(k) = 2\pi i \left\{ \int_0^1 t \hat{f}(t) e^{-2\pi i k t} \, dt + \int_0^1 (t-1) \hat{f}(t-1) e^{-2\pi i k t} \, dt \right\}$$

$$= 2\pi i \left\{ \int_0^1 t \hat{f}(t) e^{-2\pi i k t} \, dt + \int_{-1}^0 \omega \hat{f}(\omega) e^{-2\pi i k \omega} \, d\omega \right\}$$

$$= 2\pi i \cdot \int_{-1}^1 t \hat{f}(t) e^{-2\pi i k t} \, dt = \dots = f'(-k).$$

$$f(z) = \int_{-1}^1 \hat{f}(t) e^{2\pi i z t} \, dt \Rightarrow f'(z) = \int_{-1}^1 2\pi i t \hat{f}(t) e^{2\pi i z t} \, dt$$

Proof: Fix $z \in \mathbb{C}$, $|h| < 1$:

$$\frac{f(z+h) - f(z)}{h} - f'(z) = \frac{f(z+h) - f(z)}{h} - \int_{-1}^1 2\pi i t \hat{f}(t) e^{2\pi i z t} \, dt$$

Prove: $\left| \frac{\sim}{\sim} - \int \sim \, dt \right| \longrightarrow 0$ as $h \to 0$.

$$\left| \int_{-1}^1 \underbrace{\hat{f}(t) e^{2\pi i z t}}_{\text{bounded}} \underbrace{\left\{ \frac{e^{2\pi i h t} - 1}{h} - 2\pi i t \right\}}_{\text{show this goes to zero uniformly.}} \, dt \right|$$

$e^x = 1 + x + O(x^2)$, x small

So: $\left|\dfrac{e^x-1-x}{x^2}\right| \le M \Rightarrow |e^x-1-x| \le M|x|^2$

Limit + L'Hopital: $\dfrac{e^x-1}{2x}$ $\quad \dfrac{e^x}{2}$

$$\left|\int_{-1}^{1} \hat{f}(t)\, e^{2\pi i z t}\left\{\frac{e^{2\pi i h t}-1-2\pi i t h}{h}\right\} dt\right| \le C\cdot|h|$$

$|t| \le 1$ $\qquad x = 2\pi i h t$

$|h| \le 1$ $\qquad |e^{2\pi i h t}-1-2\pi i h t| \le M|2\pi i h t|^2 \le M\,4\pi^2|h|^2$

$|ht| \le 1$

So: $\hat{u}(k) = f(-k)$

$\hat{v}(k) = f'(-k)$

Fourier sum of u.

$$u_N(t) = \sum_{k=-N}^{N} \hat{u}(k)\, e^{2\pi i k t} = \sum_{k=-N}^{N} f(-k)\, e^{-2\pi i k t}$$

$$= \sum_{k=-N}^{N} f(k)\, e^{2\pi i k t}.$$

$$v_N(t) = \sum_{k=-N}^{N} f'(k)\, e^{-2\pi i k t}.$$

Now:

$$A_N := \int_{-1}^{1} \left\{ (1-|t|) \mathcal{U}_N(t) + (2\pi i)^{-1} \operatorname{sgn}(t) \mathcal{V}_N(t) \right\} e^{2\pi i z t} \, dt$$

$$= \int_{-1}^{1} (1-|t|) \mathcal{U}_N(t) e^{2\pi i z t} \, dt + (2\pi i)^{-1} \int_{-1}^{1} \operatorname{sgn}(t) \mathcal{V}_N(t) e^{2\pi i z t} \, dt$$

$$= \sum_{k=-N}^{N} f(k) \int_{-1}^{1} (1-|t|) e^{-2\pi i k t} e^{2\pi i z t} \, dt + (2\pi i)^{-1} \sum_{k=-N}^{N} f'(k) \int_{-1}^{1} \operatorname{sgn}(t) e^{-2\pi i k t} e^{2\pi i z t} \, dt$$

BREAK

$$A_N = \sum_{k=-N}^{N} f(k) \underbrace{\int_{-1}^{1} (1-|t|) e^{2\pi i t (z-k)} \, dt} + (2\pi i)^{-1} \sum_{k=-N}^{N} f'(k) \int_{-1}^{1} \operatorname{sgn}(t) e^{2\pi i t (z-k)} \, dt$$

$$= \sum_{k=-N}^{N} f(k) \left(\frac{\sin \pi (z-k)}{\pi (z-k)} \right)^2 + \sum_{k=-N}^{N} f'(k) (z-k) \left(\frac{\sin \pi (z-k)}{\pi (z-k)} \right)^2$$

$$= \sum_{k=-N}^{N} f(k) \frac{(\sin \pi z)^2}{\pi^2 (z-k)^2} + \sum_{k=-N}^{N} f'(k) \frac{(\sin \pi z)^2}{\pi^2 (z-k)}$$

$$= \underbrace{\left(\frac{\sin \pi z}{\pi} \right)^2 \left\{ \sum_{k=-N}^{N} \frac{f(k)}{(z-k)^2} + \sum_{k=-N}^{N} \frac{f'(k)}{z-k} \right\}}_{=B_N} .$$

$$A_N = B_N$$

We need to do $N \to \infty$.

$$A_N = \int_{-1}^{1} (1-|t|) u_N(t) e^{2\pi i z t} dt + (2\pi i)^{-1} \int_{-1}^{1} \operatorname{sgn}(t) v_N(t) e^{2\pi i z t} dt$$

Want to show:

$$u_N(t) \longrightarrow u(t) \quad :$$

•) u is continuous.

•) $\sum_{k\in\mathbb{Z}} |\hat{u}(k)| = \sum_{k\in\mathbb{Z}} |f(-k)| = \sum_{k\in\mathbb{Z}} |f(k)| = |f(0)| + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} |f(k)| \le |f(0)| + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{A}{|k|^{1+\delta}} < \infty$

$\downarrow$ This is absolutely convergent. (b/c $f \in M(\mathbb{R})$.)

$\Rightarrow u_N(t) \xrightarrow[N\to\infty]{} u(t)$ uniformly. (THM 2 of Fourier series.)

So: A_N :

$N \to \infty \quad A_N = \int_{-1}^{1} (1-|t|) \underline{u(t)} e^{2\pi i z t} dt + \quad \ldots ? \ldots$

Show this goes to zero:

$$\left| (2\pi i)^{-1} \int_{-1}^{1} \operatorname{sgn}(t) v_N(t) e^{2\pi i z t} dt - (2\pi i)^{-1} \int_{-1}^{1} \operatorname{sgn}(t) v(t) e^{2\pi i z t} dt \right|$$

$$\le (2\pi)^{-1} \int_{-1}^{1} \left|\operatorname{sgn}(t) \cdot e^{2\pi i z t}\right| \left|v_N(t) - v(t)\right| dt$$

$\downarrow |e^z| \le e^{|z|}$

$$\le (2\pi)^{-1}\int_{-1}^{1} 1\cdot e^{2\pi i|z|t}\,|v_N(t)-v(t)|\,dt$$

$$\le (2\pi)^{-1} e^{2\pi|z|}\int_{-1}^{1}|v_N(t)-v(t)|\cdot 1\,dt$$

Cauchy Schwarz Ineq.

$$\le (2\pi)^{-1} e^{2\pi|z|}\left\{\int_{-1}^{1}|v_N(t)-v(t)|^2\,dt\right\}^{1/2}\left\{\int_{-1}^{1}1^2\,dt\right\}^{1/2}$$

$$\le (2\pi)^{-1} e^{2\pi|z|}\sqrt{2}\underbrace{\left(\int_{-1}^{1}|v_N(t)-v(t)|^2\,dt\right)}^{1/2}$$

$\to 0$ by MEAN Square Convergence.

[Proved using vector squares.]

$\longrightarrow 0$ as $N\to\infty$.

So: As $N\to\infty$:

$$\lim_{N\to\infty} A_N = \int_{-1}^{1}\{(1-|t|)u(t)+(2\pi i)^{-1}\mathrm{sgn}(t)v(t)\}e^{2\pi i z t}\,dt$$

$$= \int_{-1}^{1} f(t)\,e^{2\pi i z t}\,dt = f(z)$$

Insert def of u and v and calculate.

$A_N = B_N$, $\lim_{N\to\infty} A_N = f(z)$

Let $\mathbb{K}$ be a compact on $\mathbb{C}$ $\Rightarrow$ $|z| \le M$ $\forall z \in K$.

$$\sum_{k=-N}^{N} \frac{f(k)}{(z-k)^2} \le \underbrace{\sum_{|k|\le M+1} \frac{f(k)}{(z-k)^2}}_{\text{finite sum}} + \underbrace{\sum_{M+1\le |k|\le N} \frac{f(k)}{(z-k)^2}}_{\text{does this converge as } N\to\infty?}.$$

$$\sum_{M+1\le|k|\le N} \frac{|f(k)|}{|z-k|^2} \le \sum_{M+1\le|k|\le N} |f(k)| \le \sum_{M+1\le|k|\le N} \frac{R}{|k|^{1+\delta}}, \quad R>0$$

$f \in M(\mathbb{R})$

$z \in \mathbb{K}$:

$$|z-k| \ge |k| - |z| \ge |k| - M \ge 1$$

T-Weierstrass

$\Rightarrow$ Converges uniformly.

CONT. NEXT TUESDAY.