(27.04.21)

Class 43: Vaaler's Formula

⊕ H. Analysis

⊕ Exercises

News:

* Seminar: WEDNESDAY 14^{15} PM — 15^{00} PM
* Don't be stressed about exam it will not be crazy
* Uploaded notes on complex analysis. website

Finishing the proof: Let $t \in [0,1]$.

$$u(t) = \hat{f}(t) + \hat{f}(t-1) \rightsquigarrow u_N(x) = \sum_{|k| \le N} \hat{u}(k) e^{2\pi i x k}$$

$$v(t) = 2\pi i \{ t \hat{f}(t) + (t-1) \hat{f}(t-1) \} \rightsquigarrow v_N(x) = \sum_{|k| \le N} \hat{v}(k) e^{2\pi i x k}$$

$$A_N(z) = \int_{-1}^{1} (1-|t|) u_N(t)\, dt + (2\pi i)^{-1} \int_{-1}^{1} \operatorname{sgn}(t) V_N(t) e^{2\pi i z t}\, dt$$

$$B_N(z) = \sum_{m=-N}^{N} \left(\frac{\sin \pi z}{\pi} \right)^2 \frac{f(m)}{(z-m)^2} + \sum_{n=-N}^{N} \left(\frac{\sin \pi z}{\pi} \right)^2 \frac{f'(n)}{(z-n)} = C_N(z) + D_N(z)$$

$$A_N(z) = B_N(z) \quad \forall n \in \mathbb{N}$$

$$\int_0^1 |v_N(t) - v(t)|^2 dt \xrightarrow[N \to \infty]{} 0$$

$$\lim_{N \to \infty} A_N(z) = \int_{-1}^{1} \{ (1-|t|) u(t) + (2\pi i)^{-1} \operatorname{sgn}(t) V(t) \} e^{2\pi i z t}\, dt$$

$$= \int_{-1}^{1} \{ \hat{f}(t) \} e^{2\pi i z t}\, dt = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i z t}\, dt$$

Fourier Inversion Formula

$$= f(z) \quad \forall z \in \mathbb{C}.$$

$\lim_{N\to\infty} B_N(z)$: $\quad C_N, D_N$ are <u>entire</u> functions. (zero of $(\sin \pi z)^2$ kills zero of $(z-m)^2$.)

Let $\mathbb{K} \subset \mathbb{C}$ be a fixed compact:

Use Weierstrass M-test:

$$* \quad \left|\left(\frac{\sin(\pi z)}{\pi}\right)^2\right| \leq P_1 \;, \; \forall z \in \mathbb{K} \qquad P_1 \text{ is a constant.}$$

also: $z \in \mathbb{K} \Rightarrow |z| \leq P_2 \quad \forall z \in \mathbb{K}.$

Reverse triangle inequality

$$|z-m| \geq |m| - |z|$$
$$\geq |m| - P_2 .$$

$$|g_m(z)| = \left|\frac{\sin \pi z}{\pi}\right|^2 \frac{|f(m)|}{|z-m|^2} \leq P_1$$

$$C_m(z) = \underbrace{\sum_{|m| \leq P_2+1} \left(\frac{\sin \pi z}{\pi}\right)^2 \frac{f(m)}{(z-m)^2}}_{\text{finite sum}} + \sum_{P_2+1 < |m| \leq N} \underbrace{\left(\frac{\sin \pi z}{\pi}\right)^2 \frac{f(m)}{(z-m)^2}}_{g_m(z)}$$

$$|g_m(z)| \leq \frac{P_1 \cdot |f(m)|}{1^2} \leq P_1 \cdot \frac{1}{|m|^{1+\delta}}$$

$$|z-m| \geq |m| - P_2 \geq P_2 + 1 - P_2 = 1.$$

$\Rightarrow$ M-test Weierstrass implies

$$\Rightarrow \sum_{P_2+1 \leq |m|} \left(\frac{\sin \pi z}{\pi}\right)^2 \frac{f(m)}{(z-m)^2} \quad \text{converges uniformly in } K.$$

$\lim_{N\to\infty} C_N(z)$ exists and converges uniformly in compacts.

$$D_N(z) = \sum_{|m| \leq P_2+1} \left(\frac{\sin \pi z}{\pi}\right)^2 \frac{f'(m)}{z-m} + \underbrace{\sum_{P_2+1 \leq |m| \leq N} \left(\frac{\sin \pi z}{\pi}\right)^2 \frac{f'(m)}{z-m}}_{S_N(z)}$$

$$\left|\frac{\sin \pi z}{\pi}\right|^2 \leq P_2 \qquad \forall z \in K.$$

$$|z-m| \geq |m| - |z| \geq |m| \geq P_2$$

Pick: $N > 2P_2$.

$N' > N$: S_N is a Cauchy sequence:

$$|S_{N'}(z) - S_N(z)| = \left| \sum_{N < |m| \leq N'} \left(\frac{\sin \pi z}{\pi}\right)^2 \frac{f'(m)}{z-m} \right|$$

$$\leq P_1 \sum_{N \leq |m| \leq N'} |f'(m)| \cdot \frac{1}{|z-m|}.$$

$$f \in \mathcal{M}(\mathbb{R}) \Rightarrow |f(x)| \leq \frac{L_1}{|x|^{1+\delta}} \quad (A)$$

We have:

$$f(z) = \int_{-1}^{1} \hat{f}(t) e^{2\pi i z t} dt \Rightarrow f'(z) = \int_{-1}^{1} \hat{f}(t)\, 2\pi i t\, e^{2\pi i z t} dt$$

$$\text{and} \quad f''(z) = \int_{-1}^{1} \hat{f}(t) (2\pi i t)^2 e^{2\pi i z t} dt$$

$$x \in \mathbb{R}: \quad |f''(x)| \leq \int_{-1}^{1} |\hat{f}(t)|\, |2\pi t|^2 dt \leq L_2 \quad (B)$$

finite integral.

Using bounding lemma on (A) and (B):

$$\Rightarrow |f'(x)| \leq \frac{L_3}{|x|^{\frac{1}{2}+\frac{\delta}{2}}}$$

$$\Rightarrow |f'(m)| \leq \frac{L_3}{|m|^{\frac{1}{2}+\frac{\delta}{2}}} \Rightarrow \sum_{|m| \leq N} |f'(m)|^2 \leq L_3^2 \sum_{|m| \leq N} \frac{1}{|m|^{1+\delta}}$$

$$\leq L_3^2\, 2 \sum_{m=1}^{\infty} \frac{1}{|m|^{1+\delta}} = L_4$$

$\Downarrow$

Comparison of series

$$\sum_{m \in \mathbb{Z}} |f'(m)|^2 = L_4 .$$

$$\leq P_1 \sum_{N \leq |m| \leq N'} |f'(m)| \cdot \frac{1}{|z-m|} .$$

$$\underset{\substack{\uparrow \\ \text{Cauchy-Schwarz inequality}}}{\le} P_1 \underbrace{\left(\sum_{N \le |m| \le N'} |f'(m)|^2 \right)^{1/2}} \left(\sum_{N \le |m| \le N'} \frac{1}{|z-m|^2} \right)^{1/2}$$

$$\le P_1 \cdot L_4^{1/2} \cdot \left(\sum_{N \le |m| \le N'} \frac{4}{|m|^2} \right)^{1/2}$$

$$|z-m| \ge |m| - P_2 \ge \frac{|m|}{2}$$

$$\Uparrow$$

$$|m| \ge 2P_2$$

$$\le 2P_1 L_4^{1/2} \underbrace{\left(\sum_{|m| \ge N} \frac{1}{|m|^2} \right)^{1/2}}_{\text{tail of a convergent series: } \to 0 \text{ as } N \to +\infty.}$$

$$\le C \varepsilon^{1/2} \text{ when } N \text{ is big.}$$

$$\exists \sum_{m \in \mathbb{Z}} a_m \Rightarrow \lim_{N \to \infty} \sum_{|m| \ge N} a_m = 0.$$

$\Rightarrow$ S_N is a uniformly Cauchy sequence.

$\Rightarrow$ limit exists: $\lim_{N \to \infty} S_N(z)$ uniformly.

D_N = finite sum + S_N $\Rightarrow$ D_N converges uniformly in $\mathbb{K}$.

$$C_N \Rightarrow B_N = C_N + D_N \Rightarrow A_N = B_N$$

$$\downarrow \qquad \downarrow$$

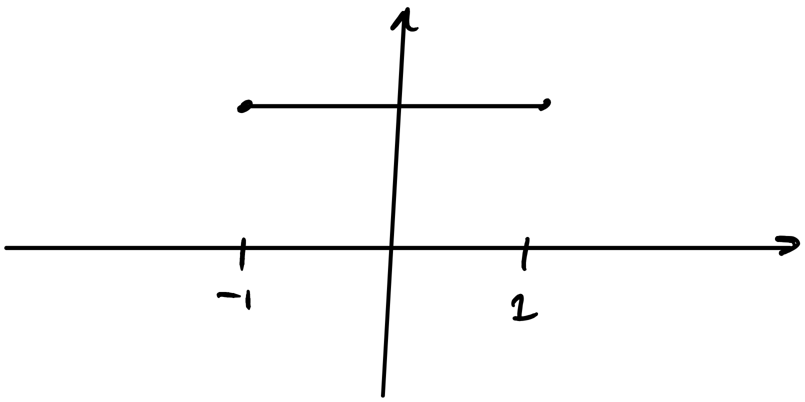
$$f(z) = \sum_{m \in \mathbb{Z}} \cdots + \sum_{m \in \mathbb{Z}} \cdots$$

Vaaler Formula

BREAK

Lebesgue Measure

$$\chi_{[-1,1]}(x) = \begin{cases} 1, & x \in [-1,1] \\ 0, & x \notin [-1,1] \end{cases}$$



$$\hat{\chi}_{[-1,1]}(t) = \int_{-\infty}^{\infty} \chi_{[-1,1]}(x) e^{-2\pi i x t} dx$$

$$= \int_{-1}^{1} 1 \, e^{-2\pi i x t} dx = \frac{e^{-2\pi i x t}}{-2\pi i t} \Bigg|_{x=-1}^{x=1} = \frac{e^{2\pi i t} - e^{-2\pi i t}}{2\pi i t}$$

$$= \frac{\sin(2\pi i t)}{\pi t}$$

L. Measure helps define Fourier transform of $f(t)$ whenever $\int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx$ exists.

$$f \in L^p(\mathbb{R}) = \{ f \text{ measurable function} : \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p} < \infty . \} \quad p \geq 1.$$

$\downarrow$ L. Measure.

Theorem: $f:[a,b]\to\mathbb{R}$ is Riemann Integrable function.

$\Rightarrow$ f is an integrable function in the Lebesgue measure on $[a,b]$.

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx.$$

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t}\,dx$$

$\hat{f}$ is continuous. $\qquad f\in M(\mathbb{R}) \Rightarrow f\in L^2(\mathbb{R})$.

$\hat{f}\to 0$, $t\to\infty$

$f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2\,dx$$

$f\in M(\mathbb{R})$, $\hat{f}\in M(\mathbb{R})$.

$f\in L^2(\mathbb{R}) \Rightarrow \hat{f}$ exists.

SHANNON FORMULA: $f\in L^2(\mathbb{R})$ $\qquad f(n)$

VAALER FORMULA: $f\in L^2(\mathbb{R})$ $\qquad f(n)$ $f'(n)$

GONCALVES FORMULA: $f\in L^2(\mathbb{R})$ $\qquad f(n)$ $f'(n)$ $f''(n)$
(2019)

(2020) BODARENKO and SEIP.

$f(\rho)$ $\quad \rho$ is a zero of $\zeta(s)$.

R.H.: $\quad \rho = \frac{1}{2} + i\gamma, \quad \gamma \in \mathbb{R}$.

Problem set 6. <u>Problem 75</u>

$$f \in M(\mathbb{R}), \quad \hat{f}(t) = 0 \quad \forall |t| \geq 1.$$

$$f(x) \geq \frac{1}{1+x^2} \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \hat{f}(0) \geq \pi \frac{e^{2\pi}+1}{e^{2\pi}-1}.$$

Solution:

First idea: $\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \underbrace{\frac{1}{1+x^2}\,dx}_{\to \arctan x} = \pi$

Not good enough!

Also have to use $\hat{f}(t) = 0 \ \forall |t| \geq 1$.

Since $f \in M(\mathbb{R})$, $\hat{f} \in M(\mathbb{R})$, we can apply PSF (Poisson Summation formula):

$$\sum_{m \in \mathbb{Z}} f(x+m) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$

for $x = 0$:

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

$$\Rightarrow \quad = \hat{f}(0) \qquad \text{b/c } \hat{f}(t) = 0 \ \forall\ |t| \geq 1.$$

So: $$\hat{f}(0) = \sum_{m\in\mathbb{Z}} f(m) \geq \sum_{m\in\mathbb{Z}} \frac{1}{1+m^2} = \pi \sum_{m\in\mathbb{Z}} \frac{1}{\pi(1+m^2)}$$

PSF

$$= \pi \sum_{k\in\mathbb{Z}} e^{-2\pi|k|}$$

← Geometric series

Recall:

$$\widehat{\frac{1}{\pi(1+x^2)}} = e^{-2\pi|k|}$$

$$= \pi\left(1 + 2\sum_{k\geq 1} e^{-2\pi k}\right)$$

$$= \pi\left(1 + 2\frac{e^{-2\pi}}{1-e^{-2\pi}}\right)$$

$\vdots$

clean up...

Possible to attain equality?

When does $$\hat{f}(0) = \pi\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right) ?$$

$\Rightarrow$ f needs to satisfy $$f(m) = \frac{1}{1+m^2} \quad \forall m \in \mathbb{Z}$$

$a_n \geq b_n$

$\Sigma a_n = \Sigma b_n$

$\Rightarrow a_n = b_n$

$(\Sigma (a_n - b_n) > 0)$

Find $f'(m)$! And use Vaaler Formula.

Let $P(x) = \frac{1}{1+x^2}$. $\Rightarrow$ $P'(x) = \frac{-2x}{(1+x^2)^2}$

$$f(x) \geq P(x) \quad \forall x \in \mathbb{R}.$$

$$f(x) - f(m) \geq P(x) - f(m) \qquad f(m) = P(m)$$

Let $x > m$: $\quad \dfrac{f(x)-f(m)}{x-m} \geq \dfrac{P(x)-f(m)}{x-m} = \dfrac{P(x)-P(m)}{x-m}$

$x \to m$: $\quad f'(m) \geq P'(m) \quad \forall m \in \mathbb{Z}$

Let now $x < m$:

$$\frac{f(x)-f(m)}{x-m} \underset{=!}{\leq} \frac{P(x)-P(m)}{x-m} \Rightarrow f'(m) \leq P'(m).$$

$$\Rightarrow f'(m) = P'(m).$$

By P.W. THM $\Rightarrow$ f is an entire function of exp. type 2π.

Now Vaaler's Formula $\Rightarrow$

$$f(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left(\overset{P(m)}{\sum_{m \in \mathbb{Z}}} \frac{1}{(1+m^2)(z-m)^2} - \overset{P'(m)}{\sum_{m \in \mathbb{Z}}} \frac{2m}{(1+m^2)(z-m)} \right).$$

Ex. 74 same idea.