CLASS 7
Convolution AND GOOD KERNELS

f, g 2π-periodic integrable

$x \in [-\pi, \pi]$

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

$f * g = g * f$ ✓

$f * (g+h) = f * g + f * h$ ✓

$(cf) * g = c(f * g) = f * (cg)$ ✓

•) $f * (g * h) = (f * g) * h$

•) $f * g$ is continuous. ✓

•) $\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n)$ ✓ $\forall n \in \mathbb{Z}$

Lemma: Let f be a function on the circle, f is integrable

$$|f(x)| \leq \underset{\sim}{B}; \quad \forall x \in [-\pi, \pi]$$

Then, there exists $\{f_K\}_{K \geq 1}$ on the circle.

i) f_K is continuous

ii) $$|f_K(x)| \leq B \quad \forall x \in [-\pi, \pi]$$

iii) $$\lim_{K \to \infty} \int_{-\pi}^{\pi} |f_K(x) - f(x)| \, dx = 0$$

f, g are 2π-periodic integrable, then $f * g$ is continuous

•) f is continuous at x_0 $(x_0 \in \mathrm{Dom} f)$

$\forall \varepsilon > 0 \ \exists \delta > 0$: $\underbrace{|x - x_0| < \delta}_{x \in \mathrm{Dom} f} \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\delta = \delta(\varepsilon; x_0)$

•) f is uniformly continuous in $\mathrm{Dom} f$

$\forall \varepsilon > 0 \ \exists \underset{\delta(\varepsilon)}{\delta} > 0$: $\underset{x, y \in \mathrm{Dom} f}{|x - y| < \delta} \Rightarrow |f(x) - f(y)| < \varepsilon$

•) $f_K \longrightarrow f$ $\left(\lim_{K \to \infty} f_K(x) = f(x) \right)$

*) Let $x \in \mathrm{Dom} f \cap f_K$: $\boxed{\lim_{K \to \infty} f_K(x) = f(x)}$

$\forall \varepsilon > 0 \ \exists K_0 \in \mathbb{N}$: $K \geq K_0 \Rightarrow |f_K(x) - f(x)| < \varepsilon$

$K_0(\varepsilon; x)$

*) $f_K \to f$ uniformly on $\mathrm{Dom} f \cap f_K$: $\forall \varepsilon > 0 \ \exists \underset{K_0(\varepsilon)}{K_0} \in \mathbb{N}$: $K \geq K_0 \Rightarrow \overline{|f_K(x) - f(x)| < \varepsilon}^{\forall x \in \mathrm{Dom} f}$

f, g : 2π-periodic - integrable

$f * g$ is continuous.

i) f, g are continuous: $f*g$ is cont.

$x_0 \in [-\pi, \pi]$; $f*g$ is cont at x_0

- $|f(x)| \leq B_1$ $\forall x \in [-\pi, \pi]$
- $|g(x)| \leq B_2$ $\forall x \in [-\pi, \pi]$
- $x_0 \in [-\pi, \pi]$
- Let $\varepsilon > 0$

$$|(f*g)(x) - (f*g)(x_0)| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\,dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x_0-y)\,dy \right|$$

$x \in [-\pi, \pi]$

$$= \frac{1}{2\pi}\left| \int_{-\pi}^{\pi} f(y)\left(g(x-y) - g(x_0-y)\right) dy \right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\,|g(x-y) - g(x_0-y)|\,dy$$

$g : [-2\pi, 2\pi] \to \mathbb{C}$, g is continuous

$\Rightarrow g$ is uniformly continuous. (exercise)

$\exists \delta > 0$: $|z - t| < \delta \Rightarrow |g(z) - g(t)| < \epsilon$

$z, t \in [-2\pi, 2\pi]$

$x \in [-\pi, \pi]$, $y \in [-\pi, \pi]$ $\Rightarrow$ $x - y \in [-2\pi, 2\pi]$, $x_0 - y \in [-2\pi, 2\pi]$

$y \in [-\pi, \pi]$

$|(x-y) - (x_0 - y)| < \delta$ $\Rightarrow$ $|g(x-y) - g(x_0-y)| < \epsilon$

$|x - x_0| < \delta$

$|x - x_0| < \delta \Rightarrow |(x-y) - (x_0 - y)| < \delta$

$\Downarrow$ UNIFORM CONT. OF g

δ

$|g(x-y) - g(x_0-y)| < \varepsilon \quad \forall y \in [-\pi, \pi]$

$$|(f*g)(x) - (f*g)(x_0)|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\underbrace{|g(x-y) - g(x_0-y)|}_{<\varepsilon} dy$$

if $|x - x_0| < \delta$

$f*g$ is uniformly continuous

$\Rightarrow f*g$ is continuous on $[-\pi,\pi]$.

$$|(f*g)(x) - (f*g)(x_0)|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\cdot \varepsilon \, dy$$

$$= \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |f(y)| dy$$

$$\leq \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} B_1 dy = \frac{\varepsilon}{\cancel{2\pi}} B_1 \cdot \cancel{2\pi}$$

$$= \varepsilon B_1$$

$\Rightarrow \forall \varepsilon > 0 \ \exists \ \delta(\varepsilon):$

$|x - y| < \delta \Rightarrow |(f*g)(x) - (f*g)(y)| < \varepsilon$

The convolution of two continuous function is a continuous function.

Let f, g 2π-periodic integrable functions.

Let $\{f_k\}_{k\geq 1}$, $\{g_k\}_{k\geq 1}$ be two sequences of cont. functions from Lemma.

$|f(x)| \leq B_1$, $\quad |g(x)| \leq B_2 \quad \forall x \in [-\pi, \pi]$

i) $|f_k(x)| \leq B_1$ $\quad$ i') $|g_k(x)| \leq B_2$

ii) $\int_{-\pi}^{\pi} |f_k(x) - f(x)|dx \xrightarrow[k\to\infty]{} 0$ $\quad$ ii') $\int_{-\pi}^{\pi} |g_k(x) - g(x)|\, dx \xrightarrow[k\to\infty]{} 0$

$f_k * g_k$ is continuous on $[-\pi, \pi]$

¿ $f * g$ is continuous?

If $\{h_k\}$ are contin. functions and $h_k \to h$ uniformly $\Rightarrow h$ is continuous.

$\{f_k * g_k\}$ are cont. functions ¿ $f_k * g_k \to f * g$ uniformly?

Let $\varepsilon > 0$:

$$(f_k * g_k)(x) - (f*g)(x) = (f_k * (g_k - g))(x) + ((f_k - f) * g)(x)$$

$$|(f_k * (g_k - g))(x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f_k(y)\,((g_k - g)(x-y))\,dy\right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_k(y)|\,|g_k(x-y) - g(x-y)|\,dy$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} B_1 |g_k(x-y) - g(x-y)|\,dy$$

$$= \frac{B_1}{2\pi}\int_{-\pi}^{\pi} |g_k(x-y) - g(x-y)|\,dy \quad x - y = z$$

$$= \frac{B_1}{2\pi}\int_{x-\pi}^{x+\pi} |g_k(z) - g(z)|\,dz$$

$\exists K_0 \in \mathbb{N}$:

$$k \geq K_0(\varepsilon) \quad = \frac{B_1}{2\pi}\int_{-\pi}^{\pi} |g_k(z) - g(z)|\,dz < \frac{B_1}{2\pi}\varepsilon$$

$\forall \varepsilon > 0 \quad \exists K_0(\varepsilon) \in \mathbb{N}: \quad K \geq K_0$

$$\Rightarrow |(f_k * g_k)(x) - (f * g)(x)|$$

$$\leq \frac{B_1}{2\pi} \cdot \varepsilon + \frac{B_2}{2\pi} \cdot \varepsilon = \underbrace{\left(\frac{B_1 + B_2}{2\pi}\right)}_{B} \varepsilon$$

$$B \cdot \varepsilon$$

$\underbrace{f_K * g_K}_{\text{continuous}} \xrightarrow[K \to \infty]{} f * g$ is uniform

$\Rightarrow f * g$ is continuous

f, g: 2π periodic-integrable function

$$\widehat{f * g}(n) = \hat{f}(n)\, \hat{g}(n) \quad \forall n \in \mathbb{Z}$$

Let f, g cont. functions ; $n \in \mathbb{Z}$

$$\widehat{f * g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f * g)(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy \right) e^{-inx}\, dx$$

Fubini's Theorem

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x-y)\, e^{-inx}\, dx \right) f(y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x-y)\, e^{-in(x-y)}\, dx \right) f(y)\, e^{-iny}\, dy$$

change of variable $x - y = z$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, e^{-inx}\, dx \right)}_{\hat{g}(n)} f(y)\, e^{-iny}\, dy$$

$$= \hat{g}(n) \cdot \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, e^{-iny}\, dy}_{} = \hat{g}(n) \cdot \hat{f}(n)$$

Let f, g : 2π - periodic integrable functions.

$$\widehat{f*g}(n) = \hat{f}(n)\,\hat{g}(n)$$

$\{f_k\}, \{g_k\}$ cont. functions as lemma.

$$\boxed{\widehat{f_k * g_k}(n) = \hat{f_k}(n) \cdot \hat{g_k}(n)} \quad \forall n \in \mathbb{Z} \quad \forall k \geq 1$$

$k \to \infty$

$$\boxed{f_k * g_k \longrightarrow f*g \text{ uniformly}}$$

$$(\widehat{f_k*g_k})(n) - (\widehat{f*g})(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f_k*g_k)(x)e^{-inx}dx - \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left((f_k*g_k)(x) - (f*g)(x)\right)e^{-inx}dx$$

$$\Rightarrow |\widehat{f_k*g_k}(n) - \widehat{f*g}(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|(f_k*g_k)(x) - (f*g)(x)|dx$$

$\forall \varepsilon > 0 \ \exists k_0 : k \geq k_0(\varepsilon) \Rightarrow |f_k*g_k(x) - f*g(x)| < \varepsilon \quad \forall x \in \mathbb{R}$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon\, dx = \varepsilon$$

$$\widehat{f_k*g_k}(n) \xrightarrow[k\to\infty]{} \widehat{f*g}(n) \checkmark$$

$$\hat{f_k}(n) - \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f_k(x) - f(x))e^{-inx}dx$$

$$|\hat{f_k}(n) - \hat{f}(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_k(x) - f(x)|dx$$

$\int_{-\pi}^{\pi}|f_k(x) - f(x)|dx \to 0 \Leftrightarrow \forall \varepsilon > 0 \ \exists k_0$

$k \geq k_0$

$$\int_{-\pi}^{\pi}|f_k(x) - f(x)|dx < \varepsilon$$

$$\Rightarrow |\hat{f_k}(n) - \hat{f}(n)| < \frac{\varepsilon}{2\pi} < \varepsilon$$

$\hat{f_k}(n) \xrightarrow[k\to\infty]{} \hat{f}(n)$ / $\hat{g_k}(n) \xrightarrow[k\to\infty]{} \hat{g}(n)$

$$\widehat{f*g}(n) = \hat{f}(n) \cdot \hat{g}(n)$$