class 8:

Good Kernels and Cesaro Summability

Let $\{K_n\}_{n\geq 1}$ be integrable functions on the circle. We say that $\{K_n\}_{n\geq 1}$ is a family of good kernels if:

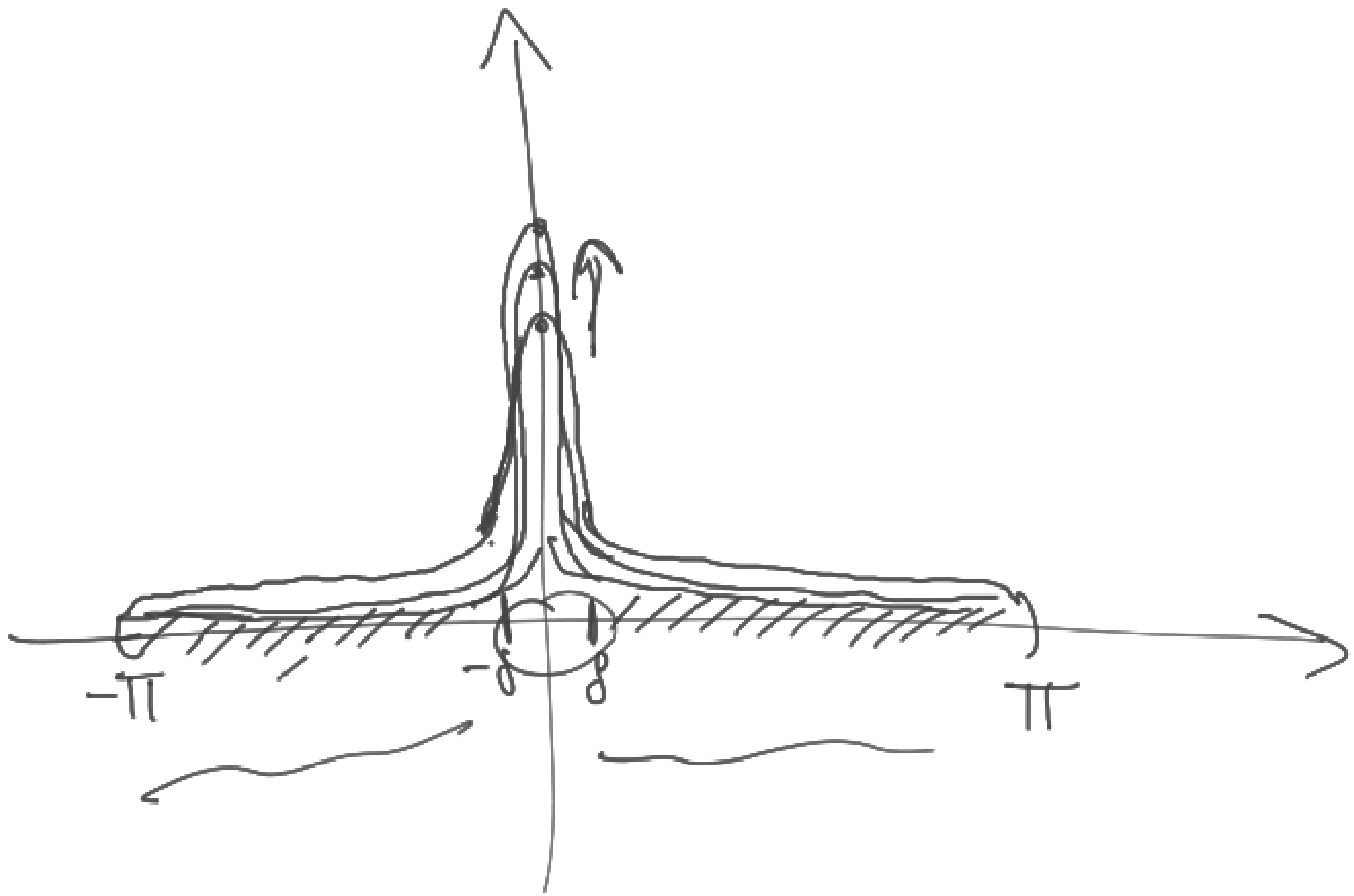
i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1 \quad \forall n\geq 1$

ii) $\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq B \quad \forall n\geq 1$ (for some $B>0$)

iii) For any $\delta>0$
$$\lim_{n\to\infty}\int_{\delta\leq|x|\leq\pi} |K_n(x)|\,dx = 0$$

Suppose that $K_n(x)\geq 0$ $\forall n\geq 1$, i), iii)

$\int_{-\pi}^{\pi} K_n(x)\,dx = 2\pi$; $\int_{\delta\leq|x|\leq\pi} K_n(x)\,dx \to 0$



Theorem: Let $\{K_n\}_{n\geq 1}$ be a family of good kernels on the circle.

i) Let f integrable on the circle:

$$(f * K_n)(x) \xrightarrow[n\to\infty]{} f(x)$$

when x is a continuity point of f.

ii) Let f continuous on the circle:

$$(f * K_n) \xrightarrow[n\to\infty]{} f$$

Uniformly

" APPROXIMATION TO THE IDENTITY "

$$(f * K_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) K_n(x-y)\,dy$$

$f(x)$ $\quad y = x$

PROOF: Let $\varepsilon > 0$, $x \in [-\pi, \pi]$; $|f(x)| \leq M$, $\forall x \in \mathbb{R}$

$$(f * K_n)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y) f(x-y)\,dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y) f(x)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\{f(x-y) - f(x)\}\,dy$$

$$|f * K_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy$$

by continuity:

$\exists\, \delta = \delta(\varepsilon, x)$: $\quad |z - x| < \delta \Rightarrow |f(z) - f(x)| < \varepsilon$

$z = x - y$ $\quad |y| < \delta \Rightarrow |f(x-y) - f(x)| < \varepsilon$

$$= \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)|\,|f(x-y) - f(x)|\,dy + \frac{1}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|\,\underbrace{|f(x-y) - f(x)|}_{\leq 2M}\,dy$$

$$\leq \frac{\varepsilon}{2\pi}\underbrace{\int_{|y|\leq\pi} |K_n(y)|\,dy}_{} + \frac{2M}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|\,dy \xrightarrow{(iii)} \quad : n \geq n_0$$

$n_0(\varepsilon; \delta; x)$

$n_0(\varepsilon, x)$

$$\frac{\varepsilon}{2\pi}\cdot B + \frac{2M}{2\pi}\cdot\varepsilon = \varepsilon\left(\frac{B}{2\pi} + \frac{2M}{2\pi}\right) = C\cdot\varepsilon$$

$\forall \varepsilon > 0$, $\overset{f \text{ is contin. a } x}{x \in [-\pi, \pi]}$ $\exists\, n_0 = n_0(\varepsilon, x)$:

$n \geq n_0 \Rightarrow |(f * K_n)(x) - f(x)| < \varepsilon$

$$\Rightarrow \lim_{n \to \infty} (f * K_n)(x) = f(x)$$

ii) We obtain: (using that f is uniform cont.)

$$|(f * K_n)(x) - f(x)| < \varepsilon \quad \forall x \in [-\pi, \pi]$$

$$\Rightarrow f * K_n \to f \text{ uniformly}$$

$S_N(f)(x) \longrightarrow f(x)$?

$S_N(f)(x) = (f * D_N)(x)$; $\quad D_N(x) = \sum_{k=-N}^{N} e^{ikx}$

$\{D_n\}_{n \geq 1}$ is a family of good kernels ?

1) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-n}^{n} e^{ikx} \right) dx$$

$$= \frac{1}{2\pi} \sum_{k=-n}^{n} \int_{-\pi}^{\pi} e^{ikx}\, dx$$

$$= \underbrace{\frac{1}{2\pi} \sum_{\substack{k=-n \\ k \neq 0}}^{n} \int_{-\pi}^{\pi} e^{ikx}\, dx}_{0} + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{0} dx$$

$$= 0 + 1 = 1$$

$\exists\, c>0:$

$$\boxed{\int_{-\pi}^{\pi} |D_n(x)|\,dx \geq c\cdot\log n} \quad \forall n\geq 2$$

EXERCISE

"$\{D_n\}_{n\geq 1}$ is not a family of GOOD KERNELS"

:(

CESÀRO SUMMABILITY:

$f * D_N \to f$?

$c_n = (-1)^n$; $n\geq 0$; $\quad S_N = \sum_{n=0}^{N} c_n = \sum_{n=0}^{N} (-1)^n$

$S_0 = 1$
$S_1 = 0$
$S_2 = 1$
$S_3 = 0$
$\vdots$

S_N is not convergent

$$\boxed{\sigma_N = \frac{S_0 + \dots + S_{N-1}}{N}}$$

$\sigma_1 = \frac{S_0}{1} = 1$

$\sigma_2 = \frac{S_0+S_1}{2} = \frac{1}{2}$

$\sigma_3 = \frac{S_0+S_1+S_2}{3} = \frac{2}{3}$

$\vdots$

$\sigma_N \longrightarrow 1/2 \quad \left(\begin{array}{l}\sigma_{2K} \to 1/2 \\ \sigma_{2K+1} \to 1/2\end{array}\right)$

σ_N: N^{th} cèsaro mean of $\{S_n\}$

σ_N: N^{th} cèsaro sum of the series $\sum_{K\geq 0} c_K$

$$\boxed{\sigma_N \xrightarrow[N\to\infty]{} \sigma}$$

$\sum_{K\geq 0} c_K$ is CÉSARO SUMMABLE TO σ.

f defined on the circle

i) $C_0(x) = \hat{f}(0)$

ii) $C_n(x) = \hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx}$

$$S_N(f)(x) = \sum_{n=0}^{N} C_n(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + S_1(f)(x) + \dots + S_{N-1}(f)(x)}{N}$$

$$= \frac{(f * D_0)(x) + (f * D_1)(x) + \dots + (f * D_{N-1})(x)}{N}$$

$$= \left(f * \frac{(D_0 + D_1 + \dots + D_{N-1})}{N}\right)(x)$$

$$= (f * F_N)(x)$$

$$\sigma_N(f)(x) = (f * F_N)(x)$$

$$F_N(x) = \frac{D_0(x) + D_1(x) + \dots + D_{N-1}(x)}{N}$$

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$$

$$F_N(x) = \frac{1}{N} \cdot \sum_{j=0}^{N-1} D_j(x) = \frac{1}{N} \underbrace{\sum_{j=0}^{N-1} \left(\sum_{k=-j}^{j} e^{ikx} \right)}$$

$\omega = e^{ix}$

$$\sum_{j=0}^{N-1} \left(\sum_{k=-j}^{j} \omega^k \right)$$

$$A = \sum_{k=a}^{b} z^k = (z^a + z^{a+1} + \dots + z^{b-1} + z^b$$

$$zA = \qquad z^{a+1} + \dots + z^{b-1} + z^b + z^{b+1}$$

$$A(1-z) = z^a - z^{b+1} \Longrightarrow A = \frac{z^{b+1} - z^a}{z-1} \quad ; \quad z \neq 1$$

$$\sum_{k=a}^{b} z^k = \frac{z^{b+1} - z^a}{z-1}$$

$\boxed{w\neq 1}$

$$\sum_{j=0}^{N-1}\left(\sum_{k=-j}^{j} w^k\right) \xrightarrow{\;w=e^{ix}\;} \qquad \sum_{k=a}^{b} z^k = \frac{z^{b+1}-z^a}{z-1},\ z\neq 1$$

$$= \sum_{j=0}^{N-1}\left\{\frac{w^{j+1}-w^{-j}}{w-1}\right\} = \frac{1}{w-1}\left(\sum_{j=0}^{N-1} w^{j+1} - \sum_{j=0}^{N-1} w^{-j}\right)$$

$$= \frac{1}{w-1}\left\{w\sum_{j=0}^{N-1} w^j - \sum_{j=0}^{N-1} \underbrace{w^{-j}}_{(w^{-1})^j}\right\} = \frac{1}{w-1}\left\{w\left(\frac{w^N-1}{w-1}\right) - \frac{(w^{-1})^N-1}{w^{-1}-1}\right\}$$

$$= \frac{1}{w-1}\left[\frac{w^{N+1}-w}{w-1} - \frac{w^{1-N}-w}{1-w}\right] = \frac{1}{w-1}\left\{\frac{w^{N+1}-w+w^{1-N}-w}{w-1}\right\} = \frac{w^{N+1}+w^{1-N}-2w}{(w-1)^2}$$

$$= \frac{w^{-1}\left(w^{N+1}+w^{1-N}-2w\right)}{\left(w^{-1/2}(w-1)\right)^2} = \frac{w^N+w^{-N}-2}{(w^{1/2}-w^{-1/2})^2} = \frac{(w^{N/2}-w^{-N/2})^2}{(w^{1/2}-w^{-1/2})^2}$$

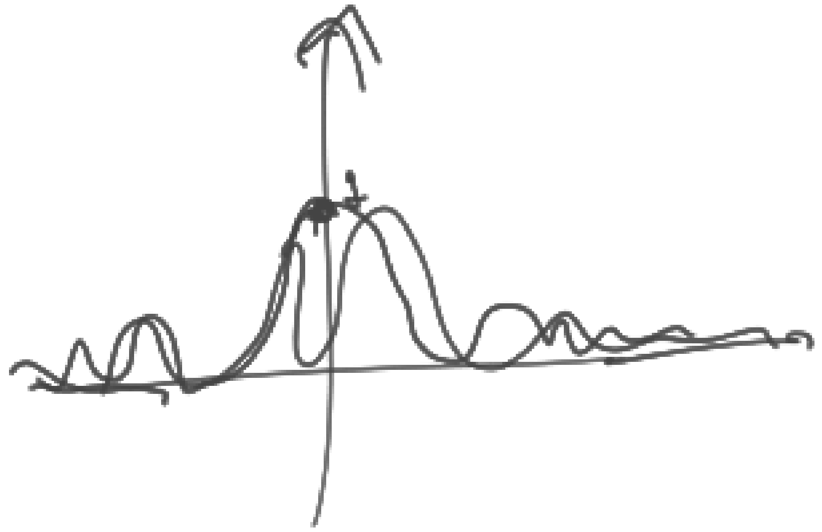
$$= \frac{\left(e^{ixN/2}-e^{-ixN/2}\right)^2}{\left(e^{ix/2}-e^{-ix/2}\right)^2} = \frac{\left(\cancel{2i}\sin(xN/2)\right)^2}{\left(\cancel{2i}\sin x/2\right)^2} = \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

$$\boxed{F_N(x) = \frac{1}{N}\cdot\frac{\sin^2(Nx/2)}{\sin^2(x/2)}} \qquad e^{ix}\neq 1$$

$$\sigma_N(f)(x) = (f * F_N)(x)$$

$$F_N(x) = \begin{cases} \dfrac{1}{N}\dfrac{\sin^2(Nx/2)}{\sin^2(x/2)} & ;\ e^{ix}\neq 1 \\ \lim\limits_{y\to x} \dfrac{1}{N}\dfrac{\sin^2(Ny/2)}{\sin^2(y/2)} & \quad e^{ix}=1 \end{cases}$$

$$\begin{cases} \left(\dfrac{\sin x}{x}\right)^2 ; & x\neq 0 \\ 1 ; & x=0 \end{cases}$$



$$F_N(x) = \boxed{\frac{1}{N}\,\frac{\sin^2(Nx/2)}{\sin^2(x/2)}}$$

$$\sigma_N(f)(x) = (f * F_N)(x)$$

*) $F_N \geq 0$

i) $$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)\,dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(D_0(x)+D_1(x)+\dots+D_{N-1}(x))\,dx}{N}$$

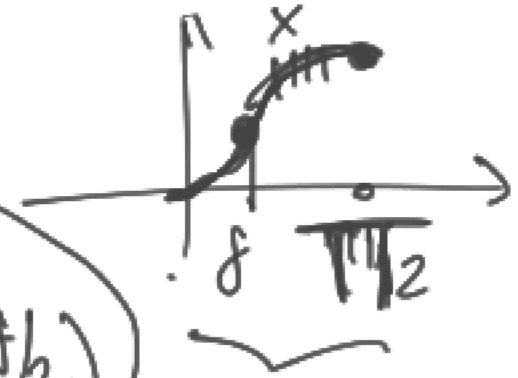
$$= 1$$

ii) $$\int_{-\pi}^{\pi} |F_n(x)|\,dx \leq 2\pi$$

iii) Let $\delta > 0$:

$$0 \leq \int_{\delta \leq |x| \leq \pi} F_N(x)\,dx$$

$$= 2\int_{\delta \leq x \leq \pi} F_N(x)\,dx \leq \frac{2}{N\sin^2(\delta/2)}\cdot\pi = \frac{2\pi}{\sin^2(\delta/2)}\cdot\frac{1}{N}$$



$$0 \leq F_N(x) \leq \frac{1}{N}\cdot\frac{1}{\sin^2(x/2)} \leq \frac{1}{N\sin^2(\delta/2)}$$

$$\sin^2(x/2) \geq \sin^2(\delta/2)$$

$\boxed{N \to \infty:}$ $$\int_{\delta \leq |x| \leq \pi} F_N(x) \xrightarrow[N\to\infty]{} 0$$

$\{F_n\}$ FAMILY OF GOOD KERNELS

Theorem:

f is integrable on the circle.

$$\sigma_N(f)(x) = f * F_N(x)$$

•) x is a continuity point of f

$$\Rightarrow \sigma_N(f)(x) \xrightarrow[N\to\infty]{} f(x)$$

the Fourier series of f is Césaro summable to f

•) f is continuous

$$\sigma_N(f)(x) \xrightarrow[\text{uniformly}]{} f(x)$$

Corollary: f is a continuous function on the circle. Let $\epsilon > 0$

$\Rightarrow \exists\, P$: trigonometric polynomial:

$$|f(x) - P(x)| < \epsilon \quad \forall x \in [-\pi, \pi]$$

$\Leftrightarrow \exists \{P_n\}$ trig. polynomials

$$\lim_{n\to\infty} P_n(x) = f(x) \text{ uniformly}$$

$$\sigma_N(f) = \frac{S_0(f) + S_1(f) + \dots + S_{N-1}(f)}{N}$$