Notes on class 5:

Corollary: Let f be a Twice continuously differentiable function on the circle. Then $\exists M>0$:

$$|\hat{f}(n)| \leq \frac{M}{|n|^2}; \quad \forall n \in \mathbb{Z}\setminus\{0\}$$

TWICE CONTINUOUSLY DIFFER. FUNCTION ON THE CIRCLE MEANS:

a) $f: \mathbb{R} \to \mathbb{C}$ 2π-periodic function, $f \in C^2(\mathbb{R})$

b) $f: [b, b+2\pi] \to \mathbb{C}$, $f(b) = f(b+2\pi)$

$f \in C^2([b, b+2\pi])$.

We have proved the corollary in the case a). Let us prove it in the case b). Assume that $f: [0, 2\pi] \to \mathbb{C}$

$f \in C^2[0, 2\pi]$. This means that $f' \in C^1[0, 2\pi]$, where $f'_+(0)$ and $f'_-(2\pi)$ are lateral derivatives.

Remembering the proof, we have obtained that:

$$\hat{f}(n) = \frac{1}{2\pi i n} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta$$

$\Rightarrow$ Integration by parts:

$$\hat{f}(n) = \frac{1}{2\pi i n} \left\{ f'(\theta) \frac{e^{-in\theta}}{-in} \Big|_0^{2\pi} - \int_0^{2\pi} f''(\theta) \left(\frac{e^{-in\theta}}{-in}\right) d\theta \right\}$$

$$= \frac{1}{2\pi i n} \left\{ \frac{f'(2\pi)}{-in} + \frac{f'(0)}{in} + \frac{1}{in} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \right\}$$

Then

$$\hat{f}(n) = \frac{f'(2\pi)}{2\pi n^2} - \frac{f'(0)}{2\pi n^2} - \frac{1}{2\pi n^2}\int_0^{2\pi} f''(\theta)e^{-in\theta}d\theta$$

Now, since that $f'(2\pi), f'(0)$ exist, (lateral derivative)

they are only constants. On another hand we had $|f''(\theta)| \leq M$

$\forall \theta \in [0, 2\pi]$

Therefore:

$$|\hat{f}(n)| \leq \left(\frac{|f'(2\pi)|}{2\pi} + \frac{|f'(0)|}{2\pi} + M\right)\frac{1}{|n|^2}$$

$$\frac{|f'(2\pi)|}{2\pi} + \frac{|f'(0)|}{2\pi} + M = N$$

$\therefore |\hat{f}(n)| \leq \frac{N}{|n|^2}$, for some $N > 0$ independent of "n".

You can apply this corollary for:

$f(\theta) = \frac{(\pi-\theta)^2}{4}$, $\theta \in [0, 2\pi]$

Note that $f''(\theta)$ exist for $\theta \in (0, 2\pi)$

and $f'_+(0) = \frac{-\pi}{2}$; and $f'_-(2\pi) = \frac{\pi}{2}$

lateral derivatives

$f'_+(0) \neq f'_-(2\pi)$