Fourier analysis - NTNU 2021
Instructor: Andrés Chirre

PROBLEM SET 1

(1) Suppose f is $2\pi-$periodic and integrable[1] on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_{a+2\pi}^{b+2\pi} f(x)\,\mathrm{d}x = \int_{a-2\pi}^{b-2\pi} f(x)\,\mathrm{d}x.$$

Also, prove that

$$\int_{-\pi}^{\pi} f(x+a)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_{-\pi+a}^{\pi+a} f(x)\,\mathrm{d}x.$$

(2) Let f be a $2\pi-$periodic and integrable function. Then, for all $n \in \mathbb{Z}$, $n \neq 0$ we have

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \big(f(x) - f(x+\pi/n)\big)e^{-inx}\,\mathrm{d}x.$$

(3) Assume that f is $2\pi-$periodic and integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \widehat{f}(0) + \sum_{n\geq 1} \big(\widehat{f}(n) + \widehat{f}(-n)\big)\cos(n\theta) + i\big(\widehat{f}(n) - \widehat{f}(-n)\big)\sin(n\theta).$$

(b) Prove that if f is even, then $\widehat{f}(n) = \widehat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\widehat{f}(n) = -\widehat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\widehat{f}(n) = 0$ for all odd n.

(e) Show that if f is real-valued, then $\overline{\widehat{f}(n)} = \widehat{f}(-n)$ for all $n \in \mathbb{Z}$.

(4) Let $f : [0, 2\pi] \to \mathbb{C}$ be a continuous function such that $f(0) = f(2\pi)$. Suppose that $\overline{\widehat{f}(n)} = \widehat{f}(-n)$ for all $n \in \mathbb{Z}$. Show that f is real-valued (i.e. $f(x) \in \mathbb{R}$ for $x \in [0, 2\pi]$).

(5) Consider the $2\pi-$periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$.

(a) Draw the graph of f.

(b) Compute the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi}\sum_{k:\,odd\geq 1} \frac{\sin(k\theta)}{k^3}.$$

(6) Let $0 < \delta \leq \pi$. On the interval $[-\pi, \pi]$ consider the function

$$f(\theta) = \begin{cases} 0 & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta & \text{if } |\theta| \leq \delta. \end{cases}$$

Thus the graph of f has the shape of a triangular tent. Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos(n\delta)}{n^2\pi\delta}\cos(n\theta).$$

[1] We refer to an integrable function $f : [a, b] \to \mathbb{C}$, when $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are Riemann integrable functions.

(7) Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$.

(a) Draw the graph of f.

(b) Compute the Fourier coefficients of f, and show that

$$\widehat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0, \\ \dfrac{-1 + (-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(c) Taking $\theta = 0$, prove that

$$\sum_{n\,:\,odd\,\geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(8) Let $\alpha \in \mathbb{C} - \mathbb{Z}$ and let f be the $2\pi-$periodic function defined by $f(\theta) = \cos(\alpha\theta)$, for $\theta \in [-\pi, \pi]$.

(a) Compute the Fourier coefficients of f.

(b) Show that the Fourier series converges[2] pointwise to $f(\theta)$ for each $\theta \in [-\pi, \pi]$.

(c) Show that

$$\frac{\alpha\pi}{\sin(\alpha\pi)} = 1 + 2\alpha^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2 - \alpha^2}.$$

(d) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)}.$$

(9) Let f be the function defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

and extend f periodically in $\mathbb{R}$.

(a) Show that

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(x)}{x}\, \mathrm{d}x.$$

(b) Show that the Fourier series converges uniformly to f.

(c) Compute

$$\lim_{N\to\infty} \int_{-N}^{N} \frac{\sin x}{x}\, \mathrm{d}x.$$

(10) (Weierstrass M-test) Let $\{f_n\}_{n\geq 1}$ be a sequence of real or complex-valued functions defined on a set $B \subset \mathbb{C}$. Suppose that there is a sequence of non-negative numbers $\{M_n\}_{n\geq 1}$ such that:

(a) $|f_n(x)| \leq M_n$, for all $x \in B$ and $n \geq 1$.

(b) $\displaystyle\sum_{n\geq 1}^{\infty} M_n < \infty$.

Then, we conclude that the series $\displaystyle\sum_{n\geq 1}^{\infty} f_n(x)$ converges absolutely and uniformly on B. A similar result holds for $\{f_n\}_{n\in\mathbb{Z}}$.

[2] The convergence of the Fourier series refers to the limit of symmetric partial sums.

(11) (Dirichlet-test) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers and $\{b_n\}_{n\geq 1}$ be a sequence of complex numbers. Suppose that

(a) a_n decreases monotonically to 0.

(b) There is $M > 0$ such that

$$\left|\sum_{n=1}^{N} b_n\right| \leq M$$

.

Then, the series $\sum_{n\geq 1}^{\infty} a_n b_n$ is convergent.

(12) Consider the function defined by

$$f(x) = \begin{cases} -\dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } -\pi \leq x < 0, \\ 0 & \text{if } x = 0, \\ \dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } 0 < x \leq \pi. \end{cases}$$

Verify that

$$f(x) \sim \frac{1}{2i}\sum_{n\neq 0} \frac{e^{inx}}{n}.$$

Show that the series converges for every $x \in [-\pi, \pi]$.

(13) Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a, b] \subset [-\pi, \pi]$.

(a) Show that the Fourier series of f is given by

$$\frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}.$$

(b) Show that if $a \neq \pi$ or $b \neq \pi$, then the Fourier series does not converge absolutely for any x. [Hint: Use the estimate $\sin^2(x) \leq |\sin(x)|$ and Dirichlet-test.]

(14) Suppose that $f : [0, 2\pi] \to \mathbb{C}$ such that $f(0) = f(2\pi)$ and $f \in C^k([0, 2\pi])$, for some $k \geq 1$. Show that there is $M > 0$ such that for all $n \geq 1$,

$$\left|\widehat{f}(n)\right| \leq \frac{M}{|n|^k}.$$

(15) Suppose that $\{f_k\}_{k\geq 1}$ is a sequence of Riemann integrable functions on the interval $[0, 2\pi]$ such that

$$\int_0^{2\pi} |f_k(x) - f(x)| \, \mathrm{d}x \to 0,$$

as $k \to \infty$. Show that $\widehat{f_k}(n) \to \widehat{f}(n)$ uniformly in n as $k \to \infty$.

Email address: carlos.a.c.chavez@ntnu.no