Fourier analysis - NTNU 2021
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PROBLEM SET 4

Let $\mathcal{M}(\mathbb{R})$ be the family of moderate decrease functions. This means that $f \in \mathcal{M}(\mathbb{R})$ if and only if $f : \mathbb{R} \to \mathbb{C}$ is a continuous function, and there are constants $\delta > 0$, $M > 0$ and $N > 0$ such that

$$|f(x)| \leq \frac{M}{|x|^{1+\delta}}$$

for $|x| \geq N$.

(45) Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function. Then, $f \in \mathcal{M}(\mathbb{R})$ if and only if there is a constant $K > 0$ such that

$$|f(x)| \leq \frac{K}{1 + |x|^{1+\delta}}$$

for all $x \in \mathbb{R}$.

(46) Let $\delta > 0$ be a real number. Then, for $f \in \mathcal{M}(\mathbb{R})$, we have

$$\delta \int_{-\infty}^{\infty} f(\delta x)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x.$$

(47) Let $f \in \mathcal{M}(\mathbb{R})$.
(a) Prove that $\widehat{f}$ is an uniformly continuous function on $\mathbb{R}$.
(b) Prove that $\widehat{f}(\xi) \to 0$, when $\xi \to \pm\infty$ (Riemann-Lebesgue Lemma).
(c) Show that if $\widehat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$, then f is identically 0.

(48) Suppose that $F(x, y)$ is a continuous function in the plane $(x, y) \in \mathbb{R}^2$. Assume there is $A > 0$ such that

$$|F(x, y)| \leq \frac{A}{(1 + |x|^{1+\delta})(1 + |y|^{1+\delta})}.$$

Define the functions

$$F_1(x) = \int_{-\infty}^{\infty} F(x, y)\, \mathrm{d}y, \quad \text{and} \quad F_2(y) = \int_{-\infty}^{\infty} F(x, y)\, \mathrm{d}x.$$

Therefore $F_1, F_2 \in \mathcal{M}(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} F_1(x, y)\, \mathrm{d}x = \int_{-\infty}^{\infty} F_2(x, y)\, \mathrm{d}y.$$

(49) Prove the following statements:
(a) If $f \in \mathcal{S}(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$, then $f * g \in \mathcal{S}(\mathbb{R})$.
(b) If $f \in \mathcal{M}(\mathbb{R})$ and $g \in \mathcal{M}(\mathbb{R})$, then $f * g \in \mathcal{M}(\mathbb{R})$.

(50) Compute the Fourier transform of the function

$$g(x) = \begin{cases} 1 + \cos x, & \text{if } |x| \leq \pi \\ 0, & \text{if } |x| > \pi. \end{cases}$$

(51) Define the function

$$f(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

(a) Compute the Fourier transform of f.

(b) Determine the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi\xi}{\pi\xi} \right)^4 d\xi.$$

(52) Let $y > 0$. Define the Poisson kernel as:

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2},$$

Prove the following identities:

(a) For all $x \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i \xi x} \, d\xi = P_y(x).$$

(b) For all $x \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} P_y(x) e^{-2\pi i \xi x} \, d\xi = e^{-2\pi|\xi|y}.$$

(53) Prove that the set $\{P_y\}_{y>0}$ is a family of good kernels as $y \to 0$.

(54) Let $a > 0$. Compute the Fourier transform of the function

$$h_a(x) = \frac{a^2 - x^2}{(a^2 + x^2)^2}.$$

(55) Prove that if $f \in \mathcal{M}(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} \, dy = 0,$$

for all $x \in \mathbb{R}$, then $f = 0$.

(56) The following exercise illustrates the principle that the decay of $\widehat{f}$ is related to the continuity properties of f.

(a) Suppose that $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f} \in \mathcal{M}(\mathbb{R})$. Prove that f satisfies a Hölder condition of order α, that is, that

$$|f(x+h) - f(x)| \leq M|h|^{\alpha},$$

for some $0 < \alpha < 1$, $M > 0$ and $x, h \in \mathbb{R}$.

(b) Let f be a continuous function on $\mathbb{R}$ which vanishes for $|x| \geq 1$, with $f(0) = 0$, and for $0 < |x| < \delta$,

$$f(x) = \frac{1}{\ln(1/|x|)},$$

for some $\delta > 0$. Prove that $\widehat{f} \notin \mathcal{M}(\mathbb{R})$.

(57) Bump functions. Examples of compactly supported functions in $\mathcal{S}(\mathbb{R})$ are very handy in many applications in analysis. Some examples are:

(a) Suppose $a < b$, and f is the function such that $f(x) = 0$ if $x \leq a$ or $x \geq b$ and

$$f(x) = e^{-1/(x-a)} e^{-1/(b-x)},$$

if $a < x < b$. Show that f is indefinitely differentiable on $\mathbb{R}$.

(b) Prove that there exists an indefinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \leq a$, $F(x) = 1$ if $x \geq b$, and F is strictly increasing on $[a, b]$.

(c) Let $\delta > 0$ be so small such that $a+\delta < b-\delta$. Show that there exists an indefinitely differentiable function g such that g is 0 if $x \leq a$ or $x \geq b$, g is 1 on $[a+\delta, b-\delta]$, and g is strictly monotonic on $[a, a+\delta]$ and $[b-\delta, b]$.

(58) Let $p, q \geq 1$ be real numbers, $\alpha, \beta, m, n \in \mathbb{Z}$ be positive numbers. Suppose that there is a constant $C > 0$ such that for any function $G \in C^\infty(\mathbb{R})$ with compact support, the following inequality holds

$$\left(\int_{-\infty}^{\infty} \left| |\xi|^\alpha \frac{d^m \widehat{G}}{d\xi}(\xi) \right|^q d\xi \right)^{1/q} \leq C \left(\int_{-\infty}^{\infty} \left| |x|^\beta \frac{d^n G}{dx}(x) \right|^p dx \right)^{1/p}.$$

Give the necessary relationship between p, q, α, β, m and n.
(Hint: Take a fixed function and its dilatations with $\delta > 0$. Taking $\delta \to 0$ and $\delta \to \infty$ one can obtain the desired relationship.)

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