Fourier analysis - NTNU 2021
Instructor: Andrés Chirre

PROBLEM SET 5

(59) Let $f \in \mathcal{M}(\mathbb{R})$ such that its Fourier transform, denoted by $\mathcal{F}(f)$, satisfies that $\mathcal{F}(f) \in \mathcal{M}(\mathbb{R})$. Prove that, for all $x \in \mathbb{R}$

$$\mathcal{F}(\mathcal{F}(f))(x) = f(-x).$$

Conclude that $\mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \circ \mathcal{F} = \mathcal{I}$ in the sense of operators, where $\circ$ is the composition of operators and $\mathcal{I}$ is the identity operator.[1]

(60) Find the constant $C > 0$ such that

$$\int_{-\infty}^{\infty} \frac{e^{-\pi x^2}}{1+x^2}\,\mathrm{d}x = C\int_{1}^{\infty} e^{-\pi x^2}\,\mathrm{d}x.$$

(61) Let $a, b > 0$. Use the Fourier transform of the function $e^{-2\pi\lambda|x|}$ to compute

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)(b^2+x^2)}\,\mathrm{d}x.$$

(62) For each $m \geq 1$ a natural number, we want to obtain a expression for

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}},$$

(a) Apply the Poisson summation formula to obtain, for $t > 0$ that:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi t|n|}.$$

(b) Prove the following identity valid for $0 < t < 1$:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}.$$

(c) Use the fact that

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}z^{2m},$$

to deduce the formula

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

The numbers B_{2m} are known as the Bernoulli numbers. Compute $\zeta(6)$ and $\zeta(8)$.

(63) The following facts have been given in class as exercises.

(a) Prove that $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t$ defines an analytic function in the semiplane $\mathrm{Re}\, s > 0$.

(b) Prove that $1/\Gamma(s)$ is an entire function. Hint: prove the fomula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

[1] $\mathcal{M}(\mathbb{R})$ denote the family of moderate decrease functions.

(c) Prove that the series $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ defines an analytic function on $\operatorname{Re} s > 0$. Show that $\eta(1) = \ln 2$. Also, show that $\eta(s)$ has zeros at the points $s = \frac{2\pi k i}{\ln 2} + 1$, where $k \in \mathbb{Z} \setminus \{0\}$.

(d) Prove that $\theta(\lambda) = \sum_{n=1}^{\infty} e^{-\pi\lambda n^2}$ is a continuous function on $(0, \infty)$.

(e) Justify the step $\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi\lambda n^2} \lambda^{s/2-1}\, \mathrm{d}\lambda = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi\lambda n^2} \lambda^{s/2-1}\, \mathrm{d}\lambda$, for $\operatorname{Re} s > 1$.

(f) Prove that $g : \mathbb{C} \to \mathbb{C}$ is an entire function, where

$$g(s) = \int_1^{\infty} \theta(u)\big(u^{-1/2-s/2} + u^{s/2-1}\big)\, \mathrm{d}u.$$

(64) Prove that, for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 0$ we have[2]

$$\sum_{n=-\infty}^{\infty} e^{-sn^2} = \sqrt{\frac{\pi}{s}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/s}.$$

(65) In class we have proved the Heisenberg uncertainty principle for $f \in \mathcal{S}(\mathbb{R})$. Which conditions are sufficient for $f \in \mathcal{M}(\mathbb{R})$ to obtain the desired inequality?

(66) Let $a, b > 0$. Let $f \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = 1$. Suppose that

$$\int_{-a}^{a} x^2 |f(x)|^2\, \mathrm{d}x \geq \frac{1}{2} \int_{-\infty}^{\infty} x^2 |f(x)|^2\, \mathrm{d}x,$$

and

$$\int_{-b}^{b} \xi^2 |\widehat{f}(\xi)|^2\, \mathrm{d}\xi \geq \frac{1}{2} \int_{-\infty}^{\infty} \xi^2 |\widehat{f}(\xi)|^2\, \mathrm{d}\xi.$$

Prove that $ab \geq 1/8\pi$.

(67) Suppose that $f \in \mathcal{M}(\mathbb{R})$ such that f and $\widehat{f}$ have compact supports. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(68) The uncertainty principle given by Donoho and Stark establishes the following: let $a, b > 0$ and $f \in \mathcal{M}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = 1$. Then

$$2(ab)^{1/2} \geq 1 - \left(\int_{|x| \geq a} |f(x)|^2\, \mathrm{d}x \right)^{1/2} - \left(\int_{|x| \geq b} |\widehat{f}(\xi)|^2\, \mathrm{d}\xi \right)^{1/2}.$$

Use this result to prove that there is a constant $C > 0$ such that

$$C \int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x \leq \int_{-\infty}^{\infty} |f(x)|^2 \left(1 - \left(\frac{\sin \pi x}{\pi x} \right)^2 \right) \mathrm{d}x.$$

for all $f \in \mathcal{M}(\mathbb{R})$ such that $\operatorname{supp} \widehat{f} \subset [-\frac{1}{2}, \frac{1}{2}]$. Hint: To inspiration, see Lemma 12 in the paper Hilbert spaces and the pair correlation of zeros of the Riemann zeta-function. Here is the arXiv link: https://arxiv.org/abs/1406.5462.

(69) Let $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f}(x) = 0$ for $|x| \geq 1$. Suppose that

$$f(x) \geq e^{-2\pi|x|}$$

for all $x \in \mathbb{R}$. Show that $\widehat{f}(0) \geq (e^{2\pi} + 1)/(e^{2\pi} - 1)$.

[2] The complex root for s is defined such that $\sqrt{1} = 1$.

Email address: `carlos.a.c.chavez@ntnu.no`