Fourier analysis - NTNU 2021
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PROBLEM SET 6

(70) Let $f \in \mathcal{M}(\mathbb{R})$ such that for all $\xi \in \mathbb{R}$ we have $|\widehat{f}(\xi)| \leq Ae^{-2\pi a|\xi|}$, for some constants $a, A > 0$. Prove that $f(x)$ is the restriction to $\mathbb{R}$ of a function $f(z)$ holomorphic in the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < b\}$ for any $0 < b < a$.

(71) Let $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f}(\xi) = O\big(e^{-2\pi a|\xi|}\big)$, for some constants $a > 0$. Suppose that $f(1/n) = 0$ for all $n \in \mathbb{N}$. Prove that $f \equiv 0$.

(72) Prove the following version of Phragmén Lindelöf Theorem: Suppose that F is a holomorphic function in the sector $\mathcal{S} = \{z \in \mathbb{C} : -\pi/4 < \arg z < \pi/4\}$ that is continuous on the closure of $\mathcal{S}$. Assume $|F(z)| \leq 1$ on the boundary of $\mathcal{S}$, and that there are constants $C, c > 0$ such that for all z in $\mathcal{S}$ we have $|F(z)| \leq Ce^{c|z|}$. Then $|F(z)| \leq 1$ for all $z \in \mathcal{S}$.

(73) Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f \in \mathcal{M}(\mathbb{R})$ and $\mathrm{supp}\, \widehat{f} \subset [-\frac{1}{2}, \frac{1}{2}]$. Suppose that $f(n) = 0$ for all $n \in \mathbb{Z} \setminus \{-1, 1\}$ and $f(-1) + f(1) = 0$.
(a) Prove that $\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = 0$.
(b) Prove that, if $f(1) = i$, then

$$f(x) = -\frac{2i\sin(\pi x)}{\pi(x+1)(x-1)}.$$

(c) Use the above fiunction to prove that

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi(x^2-1)}\right)^2 \mathrm{d}x = \frac{1}{2}.$$

(74) Let $\mathcal{F}$ be the family of functions $f : \mathbb{R} \to \mathbb{R}$ such that satisfying the following properties:
(a) $f \in \mathcal{M}(\mathbb{R})$.
(b) $f(x) \geq 0$ for all $x \in \mathbb{R}$.
(c) $f(0) = 1$.
(d) $\mathrm{supp}\, \widehat{f} \subset [-1, 1]$.
Find

$$\inf_{f \in \mathcal{F}} \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x.$$

Moreover, show that there is a unique function $f \in \mathcal{F}$ such that the above infimum is attained.

(75) Let $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f}(x) = 0$ for $|x| \geq 1$. Suppose that

$$f(x) \geq \frac{1}{1+x^2}.$$

for all $x \in \mathbb{R}$. Show that $\widehat{f}(0) \geq \pi(e^{2\pi}+1)/(e^{2\pi}-1)$. In fact, prove that there is a unique function satisfying the mentioned properties and $\widehat{f}(0) = \pi(e^{2\pi}+1)/(e^{2\pi}-1)$.

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