Fourier Analysis 2022

Professor Andrés Chirre

Wikipage

Schedule

Jamboard

Problem sets

Introduction: Fourier series

Periodic functions:

$\sin(x) = \sin(x+2\pi)$ $\quad \forall x \in \mathbb{R}$

$\cos(x) = \cos(x+2\pi)$

$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y \quad \forall x, y \in \mathbb{R}$

$\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$

$e^{ix} = \cos x + i \sin x \qquad (i^2 = -1)$

$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

$e^{2\pi i n} = (e^{2\pi i})^n = 1^n = 1$

$$\boxed{e^{2\pi i n} = 1 \quad \forall n \in \mathbb{Z}}$$

$e^{\pi i} = \cos \pi + i \sin \pi = -1 + i0 = -1$

$$\boxed{e^{\pi i n} = (-1)^n \quad \forall n \in \mathbb{Z}}$$

$e^{ix} = \cos x + i \sin x, \quad x \in \mathbb{R}$

$e^{-ix} = \cos x - i \sin x \quad \downarrow (-)$

$\dfrac{e^{ix} + e^{-ix}}{2} = \cos x$

$\cos x = \frac{1}{2} \cdot e^{-ix} + \frac{1}{2} e^{ix}$

$\sin x = -\frac{1}{2i} e^{-ix} + \frac{1}{2i} e^{ix}$

$\cos(2x) = \frac{1}{2} e^{-2ix} + \frac{1}{2} e^{2ix}$

$$\cos^2 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2$$

$$= \frac{e^{2ix} + 2 + e^{-2ix}}{4}$$

$$= \frac{1}{4}e^{-2ix} + \frac{1}{2}e^{i0} + \frac{1}{4}e^{2ix}$$

$f: [0, 2\pi] \longrightarrow \mathbb{C}$

$$f(x) = \sum_{n=-\infty}^{n=+\infty} a_n e^{inx} \; ?$$

$a_n \in \mathbb{C}$

Let $f: [a,b] \to \mathbb{C}$ be an integrable function. Define the n^{th} Fourier coefficient of f as follows:

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x) e^{\frac{-2\pi i n x}{L}} dx, \quad \text{where } L = b - a$$

for $n \in \mathbb{Z}$.

$f: [0, 2\pi] \to \mathbb{C}$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} dx$$

$f: [-\pi, \pi] \to \mathbb{C}$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\frac{2\pi i n \theta}{L}}$$

Fourier Series

It means that the Fourier series is ....

$f: [0,2\pi] \longrightarrow \mathbb{C}$

$$f(\theta)=\theta$$

Let $n \in \mathbb{Z}$:

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \theta e^{-in\theta}\, d\theta$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} \theta\, d\theta = \frac{1}{2\pi}\cdot \left\{ \frac{\theta^2}{2}\Big|_0^{2\pi} \right\}$$

$$= \frac{1}{2\pi}\left[\frac{(2\pi)^2}{2} - \frac{0^2}{2} \right] = \frac{1}{\cancel{2\pi}} \frac{\cancel{(2\pi)^2}}{2} = \pi$$

Let $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \theta e^{-in\theta}\, d\theta$$

Theorem: Let $f \in C^1[a,b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

INTEGRATION BY PARTS: Let $F, G \in C^1([a,b])$

Then:

$$(FG)' = F'G + FG'$$

$$\underbrace{\int_a^b (FG)'(x)\,dx}_{(FG)(b) - (FG)(a)} = \int_a^b F'(x) G(x)\,dx + \int_a^b F(x) G'(x)\,dx$$

$$\Rightarrow \boxed{\int_a^b F(x) G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x) G(x)\,dx}$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \theta \cdot \left(\frac{e^{-in\theta}}{-in} \right)' d\theta$$

$$= \frac{1}{2\pi}\left[\theta \frac{e^{-in\theta}}{-in}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{e^{-in\theta}}{-in}\, d\theta \right]$$

$$= \frac{1}{2\pi}\left[2\pi \cdot \frac{e^{-2\pi i n}}{-in} + \frac{1}{in}\int_0^{2\pi} e^{-in\theta}\, d\theta \right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{-in} + \frac{1}{in}\left\{ \frac{e^{-in\theta}}{-in}\Big|_0^{2\pi} \right\} \right] = \frac{1}{2\pi}\left\{ \frac{2\pi}{-in} + \frac{1}{in}\left[\cancel{\frac{1}{-in}} - \cancel{\frac{1}{-in}} \right] \right\} = -\frac{1}{in}$$

$f: [0, 2\pi] \to \mathbb{C}$

$f(\theta) = \theta$

$$\hat{f}(n) = \begin{cases} \pi & n = 0 \\ \dfrac{i}{n} & n \neq 0 \end{cases}$$

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

$g: [-\pi, \pi] \to \mathbb{C}$

$g(\theta) = \theta$

$$\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta} d\theta$$

$$\hat{g}(0) = 0 \checkmark$$

$n \neq 0$

$$\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta \left(\frac{e^{-in\theta}}{-in} \right)' d\theta$$

$$= \frac{1}{2\pi} \left[\theta \frac{e^{-in\theta}}{-in} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{-in} d\theta \right]$$

$$= \frac{1}{2\pi} \left[\pi \frac{e^{-in\pi}}{-in} - (-\pi) \frac{e^{in\pi}}{(-in)} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-in\theta} d\theta \right]$$

$$= \frac{1}{2\pi} \left[\pi \frac{(-1)^n}{-in} + \pi \frac{(-1)^n}{-in} + \frac{1}{in} \left\{ \frac{e^{-in\theta}}{-in} \Big|_{-\pi}^{\pi} \right\} \right]$$

$$= \frac{1}{2\pi} \left[-2\pi \frac{(-1)^n}{in} + \frac{1}{in} \left[\cancel{\frac{e^{-in\pi}}{-in}} - \cancel{\frac{e^{in\pi}}{-in}} \right] \right]$$

$$= \cancel{\frac{1}{2\pi}} \left[\cancel{-2\pi} \frac{(-1)^n}{in} \right] = \frac{(-1)^{n+1}}{in}$$

$$g(\theta) \sim \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \hat{g}(n) e^{in\theta} = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} e^{in\theta}$$

$$\lim_{n\to\infty} \sum_{n=1}^{N} \frac{1}{n} = \infty$$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} e^{in\theta} \qquad \theta \in [-\pi, \pi]$$