Class 4:

Theorem: Let f be a integrable function defined on the circle. Suppose that $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$. If θ_0 is a point of continuity $\Rightarrow f(\theta_0) = 0$.

Corollary 1: Let f be a continuous function on the circle. If $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$ $\Rightarrow f \equiv 0$.

Corollary 2: If f, g are two continuous functions on the circle such that $\hat{f}(n) = \hat{g}(n)$ $\forall n \in \mathbb{Z} \Rightarrow f \equiv g$

Theorem: Let f be a continuous function defined on the circle. Suppose that

$$\boxed{\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty}$$

Then:

$$S_N(f)(\theta) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\theta}$$

satisfies

$$S_N(f)(\theta) \longrightarrow f(\theta)$$

$N \to \infty$ uniformly in "θ"

$$\boxed{\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta} = f(\theta)}$$

$\forall \varepsilon > 0$ $\exists N_0$ $(N_0(\varepsilon):)$

$N \geq N_0$

$$|S_N(f)(\theta) - f(\theta)| < \varepsilon$$

$$\forall \theta \in [-\pi, \pi]$$

PROOF: Let $f:[-\pi,\pi]\to\mathbb{C}$ continuous; $f(-\pi)=f(\pi)$

Let $S_N(f)(\theta)=\sum_{k=-N}^{N}\hat{f}(k)e^{ik\theta}$, $\theta\in[-\pi,\pi]$

1) $S_N(f)$ is a continuous function in $[-\pi,\pi]$ of f

2) $S_N(f)$ satisfies $(S_N(f))(\pi)=(S_N(f))(-\pi)$.

Weierstrass-Test: Let $g_n: B\subset\mathbb{C}\to\mathbb{C}$ (or $\mathbb{R}$) be functions such that

i) $|g_n(\theta)|\le M_n$ $\forall\,\theta\in B,\ \forall\, n\ge 1$

ii) $\sum_{n=1}^{\infty}M_n<\infty$

$\Rightarrow$ $\sum_{n=1}^{N}g_n(\theta)\xrightarrow[N\to\infty]{}\sum_{n=1}^{\infty}g_n(\theta)$ uniformly on B.

Define $g_k(\theta)=\hat{f}(k)e^{ik\theta}$

$g_k:[-\pi,\pi]\to\mathbb{C}$

i) $|g_k(\theta)|\le|\hat{f}(k)|$ $\forall\,\theta\in[-\pi,\pi],\ \forall\, k\in\mathbb{Z}$

ii) $\sum_{k\in\mathbb{Z}}|\hat{f}(k)|<\infty$ ✓ (CONDITION!)

Then: (USING W.TEST)

$\sum_{k=-N}^{N}g_k(\theta)\xrightarrow[N\to\infty]{}g(\theta):=\sum_{k=-\infty}^{\infty}g_k(\theta)$ (uniformly)

$\Downarrow$ $S_N(f)(\theta)$

$\Downarrow$ $g(\theta):=\sum_{k=-\infty}^{\infty}\hat{f}(k)e^{ik\theta}$

$S_N(f)(\theta)\xrightarrow[N\to\infty]{}\sum_{k=-\infty}^{\infty}\hat{f}(k)e^{ik\theta}=:g(\theta)$

uniformly on $[-\pi,\pi]$

i) g is a continuous function on $[-\pi,\pi]$

ii) $g(\pi)=g(-\pi)\Rightarrow g$ is a cont. funct on the circle.

Let us compute $\hat{g}(n)$: For $n \in \mathbb{Z}$:

$$\hat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\lim_{N\to\infty} S_N(f)(\theta)\right) e^{-in\theta}\, d\theta$$

UNIFORM CONVERG.

$$= \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(f)(\theta)\, e^{-in\theta}\, d\theta$$

$$= \lim_{N\to\infty} \widehat{S_N(f)}(n)$$

$$= \lim_{N\to\infty} \hat{f}(n) = \hat{f}(n)$$

$\therefore\ \hat{g}(n) = \hat{f}(n) \quad \forall n \in \mathbb{Z}$

$\Rightarrow\ g \equiv f$

$S_N(f)(\theta) \xrightarrow[N\to\infty]{} f(\theta)$ uniformly in $[-\pi, \pi]$.

Example:

$f : [0, 2\pi] \to \mathbb{R}$ $\qquad f(0) = \frac{\pi^2}{4} = f(2\pi)$

$$f(\theta) = \frac{(\pi-\theta)^2}{4}$$

$$\hat{f}(n) = \begin{cases} \frac{1}{2n^2}; & n \neq 0 \\ \frac{\pi^2}{12}; & n = 0 \end{cases}$$

"compute that"

f is a continuous function defined on the circle.

$$\text{i. } \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\ ?$$

$a_n \in \mathbb{C}$

$\sum_{n\in\mathbb{Z}} a_n$ is conv. $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$, $\sum_{n=-\infty}^{0} a_n$ are convergent

•) $\sum_{n\in\mathbb{Z}} a_n$ is convergent $\Rightarrow$ $\lim_{N\to\infty} \sum_{k=-N}^{N} a_n$ exist

$\sum_{k=-\infty}^{\infty} a_n = \sum_{n\in\mathbb{Z}} a_n.$

•) $\lim_{N\to\infty} \sum_{k=-N}^{N} a_n$ can exist but no necessary $\sum_{n\in\mathbb{Z}} a_n$

If $a_n \geq 0$ $(a_n \in \mathbb{R})$

$\Rightarrow$ $\sum_{n\in\mathbb{Z}} a_n$ is convergent $\Leftrightarrow$ $\sum_{n=-\infty}^{\infty} a_n$ is convergent

$\lim_{M\to\infty} \left(\lim_{N\to\infty} \sum_{n=-N}^{M} a_n\right)$

$\lim_{N\to\infty} \left(\lim_{M\to\infty} \sum_{n=-N}^{M} a_n\right)$

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty \Leftrightarrow \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

$$\text{¿} \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \text{?}$$

$$\sum_{n=-N}^{N} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{n=1}^{N} |\hat{f}(n)| + \sum_{n=-N}^{-1} |\hat{f}(n)|$$

$$= \frac{\pi^2}{12} + 2\left(\sum_{n=1}^{N} \frac{1}{2n^2}\right)$$

$$\sum_{n=-N}^{N} |\hat{f}(n)| = \frac{\pi^2}{12} + \sum_{n=1}^{N} \frac{1}{n^2}$$

$\frac{\pi^2}{12}$ + $\sum_{n=1}^{N} \frac{1}{n^2}$, has limit when $N \to \infty$?

$$S_N = \sum_{n=1}^{N} \frac{1}{n^2}$$

$$0 \leq S_1 \leq S_2 \leq S_3 \leq S_4 \leq \cdots$$

S_N is bounded?

Let $n \geq 2$

$$n-1 \leq t \leq n$$

$$\frac{1}{n} \leq \frac{1}{t}$$

$$\frac{1}{n^2} \leq \frac{1}{t^2}$$

$$\underbrace{\int_{n-1}^{n} \frac{1}{n^2} dt}_{} \leq \int_{n-1}^{n} \frac{1}{t^2} dt$$

$$\frac{1}{n^2} \leq \int_{n-1}^{n} \frac{1}{t^2} dt$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \underbrace{\sum_{n=2}^{N} \int_{n-1}^{n} \frac{1}{t^2} dt}_{}$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \underbrace{\int_{1}^{N} \frac{1}{t^2} dt}_{}$$

$$\int_1^N t^{-2} dt$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \left.\frac{t^{-1}}{-1}\right|_1^N$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \frac{N^{-1}}{-1} - \left(\frac{1^{-1}}{-1}\right)$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq 1 - \frac{1}{N}$$

$$\Rightarrow S_N \leq 2 - \frac{1}{N} \leq 2$$

∘) S_N is increasing

∘) S_N is bounded

$\Downarrow$

$\exists \lim_{N \to \infty} S_N$

$$\sum_{n=-N}^{N} |\hat{f}(n)| = \frac{\pi^2}{12} + \sum_{n=1}^{N}\frac{1}{n^2}$$

$$\lim_{N\to\infty}\sum_{n=-N}^{N}|\hat{f}(n)| \text{ exist}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}|\hat{f}(n)| < \infty$$

$$\boxed{\sum_{n\in\mathbb{Z}}|\hat{f}(n)| < \infty}$$

BY THEOREM

$$S_N(f)(\theta) \xrightarrow[N\to\infty]{} f(\theta)$$

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\hat{f}(k)e^{ik\theta} = f(\theta) \quad \forall\theta\in[0,2\pi]$$

In particular, for the point $\theta = 0$

$$\Rightarrow \underbrace{\lim_{N\to\infty}\sum_{k=-N}^{N}\hat{f}(k)}_{} = f(0)$$

$$\lim_{N\to\infty}\left\{\frac{\pi^2}{12} + \sum_{n=1}^{N}\frac{1}{n^2}\right\} = \frac{\pi^2}{4}$$

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{n^2}\right\} = \frac{\pi^2}{4} - \frac{\pi^2}{12}$$

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{n^2}\right\} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$\sum_{n=1}^{\infty} \frac{1}{n^6}$; $\sum_{n=1}^{\infty} \frac{1}{n^4}$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\zeta(6)$ $\zeta(4)$ $\zeta(2)$

$\zeta(3)$, $\zeta(5)$, $\zeta(7)$

COROLLARY : Let f be a function defined on the circle "TWICE CONTINUOUSLY DIFFERENTIABLE."

1°) $f : \mathbb{R} \to \mathbb{C}$ 2π-periodic

$f \in C^2(\mathbb{R})$

2°) $f : [0, 2\pi] \to \mathbb{C}$; $f(0) = f(2\pi)$

$f \in C^2([0, 2\pi])$

$\Leftrightarrow f' \in C^1([0, 2\pi])$

3°) 4°)

Then : $\exists M > 0$:

$\boxed{|\hat{f}(n)| \leq \frac{M}{|n|^2}}$; for all $n \in \mathbb{Z}$, $n \neq 0$

$$\sum_{n=-N}^{N} |\hat{f}(n)| \leq \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{M}{|n|^2} + |\hat{f}(0)|$$

$$2M \sum_{n=1}^{N} \frac{1}{n^2} + |\hat{f}(0)|$$

$\Rightarrow \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$; $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$

$\Rightarrow$ $\boxed{S_N(f)(\theta) \xrightarrow{N \to \infty} f(\theta) \text{ uniformly}}$

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta} = f(\theta)$$

PROOF: Let $f : [-\pi, \pi] \to \mathbb{C}$;

$f \in C^2([-\pi,\pi])$; $\boxed{f(-\pi) = f(\pi)}$

Let $n \neq 0$:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) \cdot \left(\frac{e^{-in\theta}}{-in}\right)' d\theta$$

$$= \frac{1}{2\pi}\left[f(\theta)\frac{e^{-in\theta}}{-in}\Big|_{\theta=-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(\theta)\left(\frac{e^{-in\theta}}{-in}\right) d\theta\right]$$

$$= \frac{1}{2\pi}\left[\cancel{\frac{f(\pi)(-1)^n}{-in} - \frac{f(-\pi)(-1)^n}{-in}} + \frac{1}{in}\int_{-\pi}^{\pi} f'(\theta)\, e^{-in\theta}\, d\theta\right]$$

$$= \frac{1}{2\pi i n}\int_{-\pi}^{\pi} f'(\theta) \cdot \left(\frac{e^{-in\theta}}{-in}\right)' d\theta$$

$$= \frac{1}{2\pi i n}\left[f'(\theta)\frac{e^{-in\theta}}{-in}\Big|_{\theta=-\pi}^{\theta=\pi} - \int_{-\pi}^{\pi} f''(\theta)\frac{e^{-in\theta}}{-in}\, d\theta\right]$$

$$\hat{f}(n) = \left|\frac{1}{2\pi i n}\left[\frac{f'_-(\pi)(-1)^n}{-in} - \frac{f'_+(-\pi)(-1)^n}{-in} + \frac{1}{in}\int_{-\pi}^{\pi} f''(\theta)e^{-in\theta}\, d\theta\right]\right|$$

$|f'(\theta)| \leq B_1$, for some $B_1 > 0$ $\forall \theta \in [-\pi, \pi]$

$|f''(\theta)| \leq B_2$, for some $B_2 > 0$

$\Rightarrow$ TRIANGLE INEQUALITY

$$\Rightarrow |\hat{f}(n)| \leq \frac{1}{2\pi|n|}\left[\frac{B_1}{|n|} + \frac{B_1}{|n|} + \frac{1}{|n|}\int_{-\pi}^{\pi} B_2\, d\theta\right]$$

$$|\hat{f}(n)| \leq \frac{C}{|n|^2} \quad \forall n \neq 0 \qquad C > 0$$