Let $f: [a,b] \to \mathbb{C}$ be an integrable function ($\mathrm{Re} f$, $\mathrm{Im} f$ are integrable in Riemann's sense), and $n \in \mathbb{Z}$

$$\hat{f}(n) = \frac{1}{b-a} \int_a^b f(x)\, e^{-2\pi i n \frac{x}{L}}\, dx$$

$f: [0, 2\pi] \to \mathbb{R}$

$f(\theta) = \theta$

$g: [-\pi, \pi] \to \mathbb{R}$

$g(\theta) = \theta$

$$\hat{g}(n) = \begin{cases} 0 & ; n = 0 \\ \frac{(-1)^{n+1}}{in} & ; n \neq 0 \end{cases}$$

$$\boxed{g(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in\theta}}$$

$\sum_{n=1}^{N} a_n$ is convergent if $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exist

$\Rightarrow \sum_{n=1}^{\infty} a_n$ (I can write)

$\sum_{n=-\infty}^{\infty} a_n$ is convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$; $\sum_{n=-\infty}^{0} a_n$ are convergent.

$$a_n = \begin{cases} 1/n & ; n \neq 0 \\ 0 & ; n = 0 \end{cases}$$

$\sum_{n=-\infty}^{\infty} a_n$?

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \frac{1}{n}$$

$$S_N = \sum_{n=-N}^{N} a_n = \sum_{n=-N}^{-1} a_n + a_0 + \sum_{n=1}^{N} a_n$$

$$= \sum_{n=-N}^{-1} 1/n + \sum_{n=1}^{N} \frac{1}{n}$$

$$= -\frac{1}{N} + \frac{1}{-N+1} + \dots + \frac{1}{-2} + \frac{1}{-1} + \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{N}$$

0 0 0 0

$S_N = 0 \Rightarrow \lim_{N\to\infty} S_N = 0$

$\{a_n\}_{n \geq 1}$, $S_N = \sum_{n=1}^{N} a_n$

$\{a_n\}_{n\in\mathbb{Z}}$ $S_{N,M} = \sum_{n=-M}^{N} a_n$ | $S_N = \sum_{n=-N}^{N} a_n$

$\sum_{n=-\infty}^{\infty} a_n$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$, $\sum_{n=-\infty}^{0} a_n$ are convergent

$\sum_{n=1}^{N} a_n$; $\sum_{n=-N}^{0} a_n$ are convergent

$\sum_{n=-N}^{N} a_n$ is convergent

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in\theta}$$

SYMMETRIC PARTIAL SUMS

$$S_N(\theta) = \sum_{n=-N}^{N} \hat{g}(n) e^{in\theta}$$

$\theta \in [-\pi; \pi]$

$$S_N(\theta) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{(-1)^{n+1}}{in} e^{in\theta}$$

$\exists \lim_{N\to\infty} S_N(\theta)$?

If the limit exist, we "say" that The Fourier series is "CONVERGENT". Let $N \geq 1$ an integer

$$S_N(\theta) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in\theta} + \sum_{n=-N}^{-1} \frac{(-1)^{n+1}}{in} e^{in\theta}$$

$$= \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in\theta} + \sum_{n=1}^{N} \frac{(-1)^{-n+1}}{i(-n)} e^{i(-n)\theta}$$

$$= \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in\theta} - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in(-\theta)}$$

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} e^{in\theta} \quad . \quad \theta \in [-\pi, \pi]$$

"DIRICHLET - TEST"

$\{a_n\}_{n\geq 1} \subset \mathbb{R}$, $\{b_n\}_{n\geq 1} \subset \mathbb{C}$

i) $a_n \to 0$, monotically decreasing

ii) $\left|\sum_{n=1}^{N} b_n\right| \leq M$, for some $M>0$ for all $N \geq 1$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} a_n b_n \text{ is convergent}}$$

Fix $\theta \in [-\pi, \pi]$

$a_n = \frac{1}{n}$; $b_n = (-1)^{n+1} e^{in\theta}$

$$\boxed{R_N = \sum_{n=1}^{N} b_n = \sum_{n=1}^{N} (-1)^{n+1} e^{in\theta}}$$

$$|R_N| = \left|\sum_{n=1}^{N} (-1)^{n+1} e^{in\theta}\right| \leq \sum_{n=1}^{N} \left|(-1)^{n+1} e^{in\theta}\right| = \sum_{n=1}^{N} 1 = N.$$

" SO BAD!"

$$R_N = \sum_{n=1}^{N} (-1)^{n+1} e^{in\theta}$$

$$(-1) e^{i\theta} R_N = \sum_{n=1}^{N} (-1)^{n+1+1} e^{i(n+1)\theta}$$

$$= \sum_{n=1}^{N} (-1)^{(n+1)+1} e^{i(n+1)\theta}$$

$$= \sum_{n=2}^{N+1} (-1)^{n+1} e^{in\theta} = \sum_{n=1}^{N+1} (-1)^{n+1} e^{in\theta} - (-1)^{1+1} e^{i\theta}$$

$$= \sum_{n=1}^{N} (-1)^{n+1} e^{in\theta} + (-1)^{N+1+1} e^{i(N+1)\theta} - e^{i\theta}$$

$$-e^{i\theta} R_N = R_N + (-1)^N e^{i(N+1)\theta} - e^{i\theta}$$

$$\Rightarrow e^{i\theta} - (-1)^N e^{i(N+1)\theta} = R_N (1 + e^{i\theta})$$

$e^{i\theta} - (-1)^N e^{i(N+1)\theta} = R_N(1+e^{i\theta})$

Suppose That $\theta \in (-\pi, \pi)$

$$R_N = \frac{e^{i\theta} - (-1)^N e^{i(N+1)\theta}}{1+e^{i\theta}}$$

$$\Rightarrow |R_N| \leq \frac{|e^{i\theta}| + |(-1)^N e^{i(N+1)\theta}|}{|1+e^{i\theta}|}$$

$$\Rightarrow |R_N| \leq \frac{2}{|1+e^{i\theta}|}$$

$$\left|\sum_{n=1}^{N} b_n\right| \leq \frac{2}{|1+e^{i\theta}|} \Rightarrow \text{"}M\text{"}$$

$\Rightarrow$ By Dirichlet - Test

$$\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} e^{in\theta} \text{ is convergent}$$

$\theta \in (-\pi, \pi)$

$$S_N(\theta) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in\theta} - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in(-\theta)}$$

When $\theta \in (-\pi, \pi)$

$\Rightarrow S_N(\theta)$ is convergent $\left(\lim_{N\to\infty} S_N(\theta) \text{ exist}\right)$

$\sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in\theta}$ is convergent $\theta \in (-\pi, \pi)$

When $\theta = \pi$ or $\theta = -\pi$

$$S_N(\pi) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in\pi} - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{in(-\pi)}$$

$$= \sum_{n=1}^{N} \frac{(-1)^{n+1}(-1)^n}{in} - \sum_{n=1}^{N} \frac{(-1)^{n+1}(-1)^n}{in} = 0$$

$S_N(-\pi) = 0 \Rightarrow \lim_{N\to\infty} S_N(\pi) = 0$

or $-\pi$

Let $N \geqslant 0$. Define

$$P_N : [-\pi, \pi] \to \mathbb{C}$$

$$P_N(\theta) = \sum_{m=-N}^{N} a_m e^{im\theta} \; ; \quad a_m \in \mathbb{C}$$

TRIGONOMETRIC POLYNOMIAL.

Let $n \in \mathbb{Z}$

$$\hat{P}_N(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_N(\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-N}^{N} a_m e^{im\theta}\right) e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-N}^{N} a_m \int_{-\pi}^{\pi} e^{im\theta} e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-N}^{N} a_m \int_{-\pi}^{\pi} e^{i(m-n)\theta}\, d\theta$$

$a_m = 1 \Rightarrow$ " P_N is THE DIRICHLET KERNEL "

Suppose that $|n| > N$

$$\hat{P}_N(n) = \frac{1}{2\pi}\sum_{m=-N}^{N} a_m \left(\frac{e^{i(m-n)\theta}}{i(m-n)} \right|_{\theta=-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\sum_{m=-N}^{N} a_m \left\{ \frac{e^{i(m-n)\pi}}{i(m-n)} - \frac{e^{i(m-n)(-\pi)}}{i(m-n)} \right.$$

$(-1)^{m-n}$ $\quad$ $(-1)^{m-n}$

$$\hat{P}_N(n) = 0.$$

Suppose that $|n| \leq N$

$$\hat{P}_N(n) = \frac{1}{2\pi}\left(a_n \int_{-\pi}^{\pi} e^{i(n-n)\theta}\, d\theta + \sum_{\substack{m=-N \\ \boxed{m \neq n}}}^{N} a_m \left[\int_{-\pi}^{\pi} e^{i(m-n)\theta}\, d\theta\right] \right)$$

$$= a_n + 0$$

$$\Rightarrow \hat{P}_N(n) = a_n$$

$$P_N(\theta) \sim \sum_{n=-N}^{N} a_n e^{in\theta}$$

$=$

$P_N: \mathbb{R} \to \mathbb{C}$

$$P_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

$$P_N(\theta+2\pi) = \sum_{n=-N}^{N} a_n e^{in(\theta+2\pi)} = \sum_{n=-N}^{N} a_n e^{in\theta} \cdot e^{in \cdot 2\pi}$$

$$= \sum_{n=-N}^{N} a_n e^{in\theta} = P_N(\theta) \quad \forall \theta \in \mathbb{R}$$

P_N is a 2π-periodic function

$g: \mathbb{R} \to \mathbb{C}$

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in(\theta+2\pi)} = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in\theta}$$

$$g(\theta+2\pi)$$

If $f: \mathbb{R} \to \mathbb{C}$

$$f(x) = f(x+2\pi) \quad \forall x \in \mathbb{R}$$

We say that f is 2π-periodic function

Let f be a 2π-periodic function integrable

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

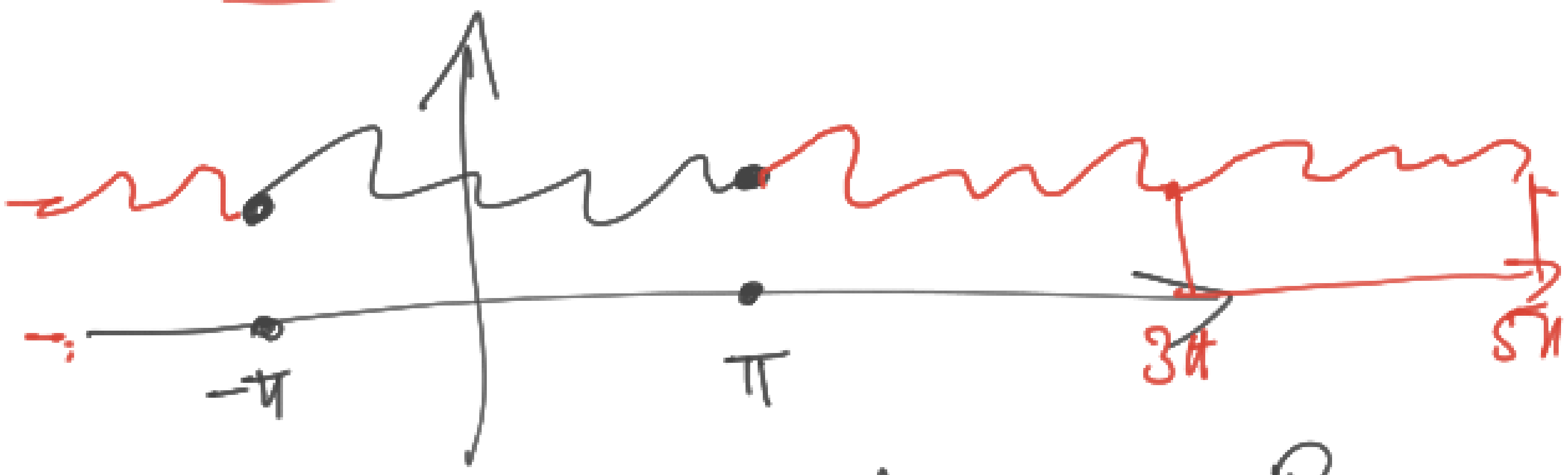
$$= \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) e^{-in\theta} d\theta$$

YOU CAN PROVE THAT

$$= \frac{1}{2\pi} \int_{a}^{a+2\pi} f(\theta) e^{-in\theta} d\theta, \quad a \in \mathbb{R}$$

b) Let $f: [-\pi, \pi] \to \mathbb{C}$

if $\boxed{f(-\pi) = f(\pi)}$ ✓



We can extend this function to $\mathbb{R}$

$f: \mathbb{R} \to \mathbb{C}$ such that f is 2π-periodic

$$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$$

$\boxed{f(0) = f(2\pi)}$

c) $f: [0, 2\pi] \to \mathbb{C}$ and we can extend to $\mathbb{R}$

d) $f: [0, a+2\pi] \to \mathbb{C}$ and we can extend to $\mathbb{R}$.

$a \in \mathbb{R}$

$\boxed{f(a) = f(a+2\pi)}$

f IS A FUNCTION DEFINED ON THE CIRCLE

if f satisfies a), b), c) or d).