Class 3 :

Let $f : [-\pi, \pi] \to \mathbb{C}$, integrable function. For $n \in \mathbb{Z}$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$f \sim \underbrace{\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}}_{\text{Fourier Series}}$$

means this is the Fourier series

$f : [-\pi, \pi] \to \mathbb{R}$

$f(\theta) = \theta$

$$\hat{f}(n) = \begin{cases} \frac{(-1)^{n+1}}{in} & ; n \neq 0 \\ 0 & n = 0 \end{cases}$$

$$f(\theta) \sim \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} e^{in\theta}$$

$f : [-\pi, \pi] \to \mathbb{C}$

$$f(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$ TRIGONOMETRIC POLYNOMIAL

$N \geq 1$; $a_n \in \mathbb{C}$

$$\boxed{\hat{f}(n) = a_n \quad \forall n \in \mathbb{Z}}$$

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{i n(\theta + 2\pi)} = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{in\theta}$$

FUNCTIONS DEFINED ON THE CIRCLE

- $f : \mathbb{R} \to \mathbb{C}$; $\boxed{f(x) = f(x + 2\pi)} \; \forall x \in \mathbb{R}$
- $f : [0, 2\pi] \to \mathbb{C}$, $f(0) = f(2\pi)$

  $f : \mathbb{R} \to \mathbb{C}$
- $f : [-\pi, \pi] \to \mathbb{C}$: $f(-\pi) = f(\pi)$
- $f : [a, a+2\pi] \to \mathbb{C}$; $f(a) = f(a+2\pi)$

$f : [-\pi, \pi] \to \mathbb{C}$, integrable, 2π-periodic

$$S_N(f)(\theta) = \underbrace{\sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}}_{\text{TRIGONOMETRIC POLYNOMIAL}}$$

WHEN $S_N(f)(\theta) \xrightarrow[N\to\infty]{} f(\theta)$? POINTWISE

UNIQUENESS OF THE FOURIER SERIES:

Theorem 1. Let f be a integrable function on the circle, such that $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$. Then, if θ_0 is a point of continuity, we have

$$f(\theta_0) = 0$$

Corollary 1: Let f be a continuous function on the circle. Suppose that $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$

$$\Rightarrow f \equiv 0$$

Corollary 2: Let f, g two continuous functions defined on the circle such that

$$\hat{f}(n) = \hat{g}(n) \quad \forall n \in \mathbb{Z} \Rightarrow f \equiv g$$

PROOF: Define $h = f - g$

$$\hat{h}(n) = \hat{f}(n) - \hat{g}(n) = 0 \quad \forall n \in \mathbb{Z}$$

$$\Rightarrow h \equiv 0 \Rightarrow f \equiv g$$

PROOF THEOREM 1:

Suppose that you have proved Theorem 1, assuming that $f : D \to \mathbb{R}$. In the general case; if $f : D \to \mathbb{C}$

$$\left.\begin{aligned} f &= u + iv \\ \bar{f} &= u - iv \end{aligned}\right\} \quad \begin{aligned} &u, v : D \to \mathbb{R} \\ &u = (f + \bar{f})/2 \\ &v = (f - \bar{f})/2i \end{aligned}$$

$$u = \frac{f + \overline{f}}{2}$$

$$\hat{u}(n) = \frac{\hat{f}(n) + \widehat{\overline{f}}(n)}{2}$$

$$\left(\widehat{\overline{f}}\right)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\overline{f})(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\, e^{-inx}\, dx$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx}$$

$$= \overline{\hat{f}(-n)}$$

$$\hat{u}(n) = \frac{\hat{f}(n) + \hat{f}(-n)}{2} = \frac{0+0}{2} = 0$$

$\hat{u}(n) = 0 \quad \forall n \in \mathbb{Z}$

u is defined on the circle (integrable)

u is continuous at θ_0

$u : D \to \mathbb{R}$

$\Rightarrow u(\theta_0) = 0$. The same for $v(\theta_0) = 0$

$\Rightarrow f(\theta_0) = 0$ (we are done with our condition).

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- $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ integrable ($|f(x)| \leq B$ $\forall x \in [-\pi,\pi]$)
- f is continuous at $\theta_0 = 0$.
- $\boxed{\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}}$

Assume that $f(0) \neq 0$. Then $f(0) > 0$ or $f(0) < 0$

Assume that $f(0) > 0$ (if not do the same proof with $-f$)

$$(*) \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = 0 \quad \forall n \in \mathbb{Z}$$

$\int_{-\pi}^{\pi} f(x) P(x)\, dx = 0$, where P is a trigonometric polynomial.

a) Since $f(0) > 0$, f is continuous at "0",

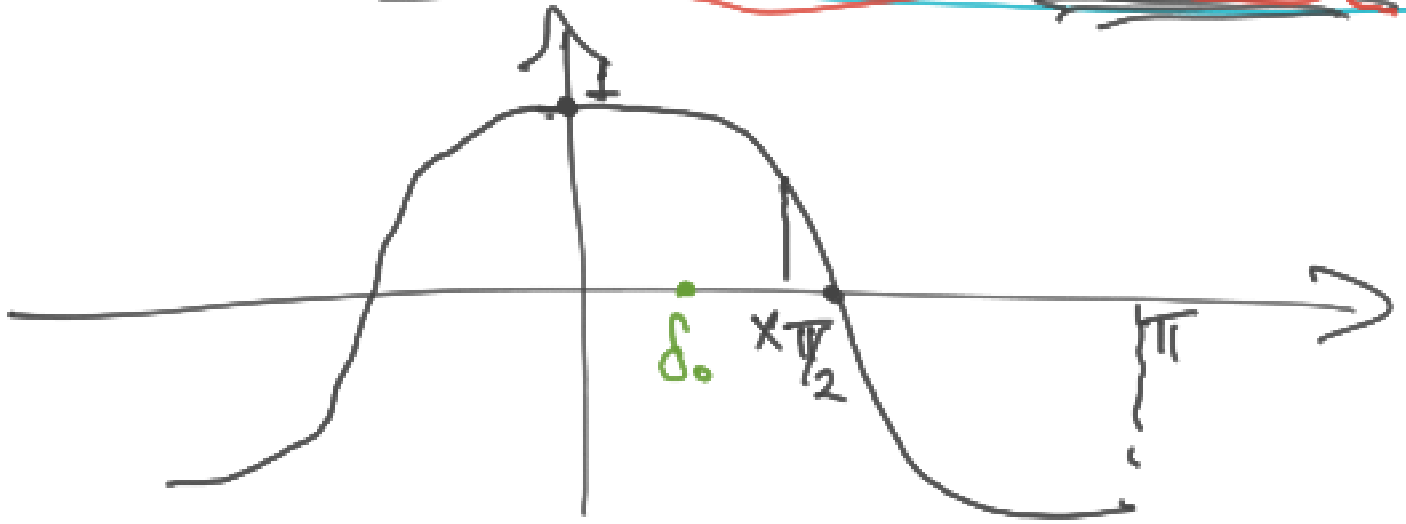
Then for $\varepsilon = f(0)/2$ $\exists\ \delta_0 > 0$:

$$|x| \le \delta_0 \Rightarrow |f(x) - f(0)| < f(0)/2$$

$$-f(0)/2 < f(x) - f(0) < f(0)/2$$

$$f(0)/2 < f(x)$$

$$\Rightarrow \boxed{\text{if } |x| \le \delta_0 < \tfrac{\pi}{2} : \ f(x) > f(0)/2}$$



Let $\delta_0 \le x \le \pi$

$$\cos \delta_0 \ge \cos x$$

$$-\cos \delta_0 \le -\cos x$$

$$1 - \cos \delta_0 \le 1 - \cos x$$

$$\varepsilon_0 \le 1 - \cos x$$

$$\boxed{\cos x + 2\frac{\varepsilon_0}{3}} \le \underbrace{1 - \frac{\varepsilon_0}{3}}$$

$$P(x) \le 1 - \frac{\varepsilon_0}{3}$$

$\forall\ x \in [\delta_0, \pi]$

$$P(x) \le 1 - \frac{\varepsilon_0}{3} \ : \ \delta_0 \le |x| \le \pi$$

$$|P(x)| \le 1 - \frac{\varepsilon_0}{3} \iff \frac{\varepsilon_0}{3} - 1 \le P(x) \le 1 - \frac{\varepsilon_0}{3}$$

$$\iff \frac{\varepsilon_0}{3} - 1 \le \cos x + 2\frac{\varepsilon_0}{3} \iff -\frac{\varepsilon_0}{3} \le \cos x + 1$$

$|P(x)| \le 1 - \frac{\varepsilon_0}{3}$; for all $\boxed{\delta_0 \le |x| \le \pi}$

$\boxed{P_k(x) = (P(x))^{2k}}$, $k \geq 1$

$0 \le P_k(x) \le \left(1 - \frac{\varepsilon_0}{3}\right)^{2k} \quad \forall k \geq 1$

$\lim\limits_{x \to 0} \cos x = 1 \Rightarrow$ For $\frac{\varepsilon_0}{3} > 0 \quad \exists\, \eta_0 > 0$

such that : $|x| \le \eta_0 < \delta_0 \Rightarrow |\cos x - 1| < \varepsilon_0/3$

$1 - \cos x < \varepsilon_0/3$

$1 - \varepsilon_0/3 < \cos x$

$1 + \varepsilon_0/3 < \cos x + 2\varepsilon_0/3$

$1 + \varepsilon_0/3 < P(x)$

$$\boxed{P_k(x) > \left(1 + \frac{\varepsilon_0}{3}\right)^{2k} \quad \forall k \in \mathbb{Z} \quad \text{for } |x| \le \eta_0 < \delta_0}$$

$$\int_{-\pi}^{\pi} f(x) P_k(x)\, dx = 0$$

$$\int_{|x| \le \eta_0} f(x) P_k(x)\, dx + \int_{\eta_0 < |x| \le \delta_0} f(x) P_k(x)\, dx + \int_{|x| > \delta_0} f(x) P_k(x)\, dx = 0$$

$\therefore \quad I_1 + I_2 + I_3 = 0$

$$I_2 : \int_{\eta_0 \le |x| < \delta_0} f(x) P_K(x)\, dx \geq 0$$

$\geq 0 \quad > 0$

$$|I_3| = \left| \int_{\delta_0 \le |x| \le \pi} f(x) P_K(x)\, dx \right|$$

$$\le \int_{\delta_0 \le |x| \le \pi} |f(x)| |P_K(x)|\, dx$$

$$\le B \left(1 - \frac{\varepsilon_0}{3}\right)^{2K} \int_{\delta_0 \le |x| \le \pi} 1\, dx$$

$$|I_3| \le B \left(1 - \frac{\varepsilon_0}{3}\right)^{2K} \cdot 2\pi$$

$|x| \le \eta_0 < \delta_0$

$$I_1 : \int_{|x| \le \eta_0} f(x) P_K(x)\, dx$$

$$\geq f(0)/2 \left(1 + \frac{\varepsilon_0}{3}\right)^{2K} \int_{|x| \le \eta_0} 1\, dx$$

$$= \frac{f(0)}{2} \left(1 + \frac{\varepsilon_0}{3}\right)^{2K} \cdot 2\eta_0$$

$$I_1 + I_2 + I_3 = 0$$

$$0 = I_1 + I_2 + I_3 \geq I_1 + I_3$$

$$\frac{f(0)}{2} \left(1 + \frac{\varepsilon_0}{3}\right)^{\boxed{2K}} 2\eta_0 \le I_1 \le -I_3 \le |I_3|$$

$$\le B \cdot 2\pi \left(1 - \frac{\varepsilon_0}{3}\right)^{\boxed{2K}}$$

$K \to \infty \quad \uparrow \infty \qquad \downarrow 0$

CONTRADICTION!

Theorem: Let f be a continuous function defined on the circle. Suppose that

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$$

$$\Rightarrow \quad S_N(f)(\theta) \xrightarrow{N\to\infty} f(\theta)$$

uniformly on the interval: "θ"

$$\sum_{n=-N}^{N} \hat{f}(n) e^{in\theta} \longrightarrow f(\theta)$$

$$\boxed{\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta} = f(\theta)}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \frac{\pi^2}{6}\ !$$