(5) $f(x) = \overline{f(x)} \quad \forall x \in [0, 2\pi]$

$G(x) = f(x) - \overline{f(x)}$

$G : [0, 2\pi] \to \mathbb{C}$

$G(x) = f(x) - \overline{f(x)}$

$G(0) = f(0) - \overline{f(0)}$

$G(2\pi) = f(2\pi) - \overline{f(2\pi)}$

$G(0) - G(2\pi) = \underbrace{f(0) - f(2\pi)}_{0} - \underbrace{\overline{\left(f(0) - f(2\pi)\right)}}_{0}$

$G(0) = G(2\pi)$

G is continuous.

Let $n \in \mathbb{Z}$

$\hat{G}(n) = \hat{f}(n) - \left(\widehat{\overline{f}}\right)(n)$

$= \hat{f}(n) - \overline{\hat{f}(-n)}$

$= \hat{f}(n) - \hat{f}(n) = 0$

$\hat{G}(n) = 0$

$\Longrightarrow$ By Theorem in class

$G \equiv 0$

$\Rightarrow f(x) = \overline{f(x)} \quad \forall x \in [0, 2\pi)$

② $f : \mathbb{R} \to \mathbb{C}$ 2π- periodic and integrable

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(f(x) - f\left(x+\frac{\pi}{n}\right)\right)e^{-inx}\,dx = \underbrace{\frac{1}{4\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx}_{} - \underbrace{\frac{1}{4\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n}\right)e^{-inx}\,dx}_{\mathrm{I}}$$

$$\mathrm{I} = \frac{1}{4\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n}\right)e^{-inx}\,dx \qquad \boxed{x+\frac{\pi}{n} = w} \quad \text{change of variable}$$

$$= \frac{1}{4\pi}\int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f(w)\,e^{-in\left(w-\frac{\pi}{n}\right)}\,dw$$

$$= \frac{1}{4\pi}\int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f(w)\,e^{-inw}\,e^{i\pi}\,dw = \frac{-1}{4\pi}\int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f(w)\,e^{-inw}\,dw$$

PROBLEM (1)

$$= \frac{-1}{4\pi}\int_{-\pi}^{\pi} f(w)\,e^{-inw}\,dw$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx \times 2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx = \hat{f}(n)$$

3.c

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, e^{-inx}\, dx + \int_{0}^{\pi} f(x)\, e^{-inx}\, dx\right]$$

$w = x+\pi$

$$= \frac{1}{2\pi}\left[\int_{0}^{\pi} f(w-\pi)\, e^{-in(w-\pi)}\, dw + \text{———}\right]$$

$$= \frac{1}{2\pi}\left[\left(\int_{0}^{\pi} f(w)\, e^{-inw}\, dw\right) e^{in\pi} + \int_{0}^{\pi} f(x)\, e^{-inx}\, dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\int \qquad\right) \cdot \left(e^{in\pi} + 1\right)\right]$$

0 when n is odd.

① ¿ $\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx$?

$x+a=w$

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi+a}^{\pi+a} f(w)\,dw = I$$

$a=0$ ✓

$a>0$:

Assume that $\boxed{-\pi \le -\pi+a \le \pi}$

$$I = \int_{-\pi+a}^{\pi} f(w)\,dw + \int_{\pi}^{\pi+a} f(w)\,dw = \int_{-\pi+a}^{\pi} f(w)\,dw + \int_{\pi}^{\pi+a} f(w-2\pi)\,dw$$

2π-periodic

$w-2\pi=\theta$

$$= \underline{\int_{-\pi+a}^{\pi} f(w)\,dw} + \underline{\int_{-\pi}^{-\pi+a} f(\theta)\,d\theta} = \int_{-\pi}^{\pi} f(x)\,dx$$