Fourier analysis - NTNU 2022
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PROBLEM SET 2

(19) Let $f:\mathbb{R}\to\mathbb{C}$ be a $2\pi-$periodic and integrable function and $k\in\mathbb{Z}$. Define the function $g(x) = e^{ikx}f(x)$. Prove that $\widehat{g}(m)=\widehat{f}(m-k)$ for all $m\in\mathbb{Z}$.

(20) Let $f:\mathbb{R}\to\mathbb{C}$ be a $2\pi-$periodic function such that $f\in C^k(\mathbb{R})$ for some $k\geq 1$. Then

$$\widehat{f^{(k)}}(n)=(in)^k\widehat{f}(n),$$

for all $n\in\mathbb{Z}$.

(21) Suppose that f,g and h are $2\pi-$periodic and integrable functions[1]. Prove that

$$(f*g)*h=f*(g*h).$$

(22) Let $f:\mathbb{R}\to\mathbb{C}$ be a $2\pi-$periodic function and P be a trigonometric polynomial. Define the function

$$R_N(\theta)=\sum_{|n|\leq N}\widehat{f}(n)\widehat{P}(n)\,e^{in\theta}.$$

Prove that R_N converges uniformly to $f*P$ in $[-\pi,\pi]$.

(23) Let P be a trigonometric polynomial of degree $N>0$. Show that P has at most $2N$ zeros. Construct a trigonometric polynomial with exactly $2N$ zeros.

(24) The function $P_r(\theta)$, called the Poisson kernel, is defined for $\theta\in[-\pi,\pi]$ and $0\leq r<1$ by the series:

$$P_r(\theta)=\sum_{n\in\mathbb{Z}}r^{|n|}e^{in\theta}.$$

(a) Prove that this series converges absolutely and uniformly.
(b) Show the formula

$$P_r(\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2}.$$

(c) Prove that for all $0\leq r<1$ we have:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta)=1.$$

(d) Prove that for every $\delta>0$,

$$\lim_{r\to 1^-}\int_{\delta\leq|x|\leq\pi}P_r(\theta)\,\mathrm{d}\theta=0.$$

(e) Conclude that for any sequence $0\leq r_n<1$ such that $r_n\to 1$ as $n\to\infty$, we have that $\{P_{r_n}\}_{n\geq 1}$ is a family of good kernels.

(25) A known-result establishes that the continuous functions on the circle can be uniformly approximated by trigonometric polynomials.

[1] We refer as an integrable function to a Riemann integrable function on any compact interval.

(a) Use the above mentioned result to prove the following statement: If f is a continuous function on the circle, and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

(b) Give an example of a function f on the circle, such that $\widehat{f}(2n) = 0$ for all $n \in \mathbb{Z} - \{0\}$ and f is not a trigonometric polynomial.

(26) Let f be a integrable function on the circle and $p \geq 1$. Prove that there is a sequence of trigonometric polynomials $\{P_k\}_{k\geq 1}$ such that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \left|P_k(x) - f(x)\right|^p \mathrm{d}x \to 0$$

(27) Prove Weierstrass approximation theorem: Let f be a continuous function defined on the interval $[a, b] \subset \mathbb{R}$. Then, for any $\varepsilon > 0$, there exists a polynomial P such that

$$|f(x) - P(x)| < \varepsilon, \quad \text{for } x \in [a, b].$$

Use this theorem to prove the following statement: If $f : [a, b] \to \mathbb{R}$ is a continuous function and

$$\int_a^b f(x)\, x^n \,\mathrm{d}x = 0,$$

for all integer $n \geq 0$, then $f \equiv 0$.

(28) Let $N \geq 1$ be a natural number and F_N be the Fejér Kernel. Prove that for $x \in \mathbb{R}$, we have

$$F_N(x) = \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) e^{ikx}.$$

(29) (de la Vallée Poussin kernel) Define the function

$$V_N(x) = 2F_{2N+1}(x) - F_N(x),$$

where F_N is the Fejér Kernel.

(a) Show that the sequence $\{V_N\}$ is an approximation of the identity.

(b) Compute $\widehat{V_N}(m)$ when $|m| \leq N + 1$ and when $|m| \geq 2N + 2$.

(30)* Let $N \geq 1$ be a natural number, D_N be the Dirichlet kernel

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta},$$

and define

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)|\, \mathrm{d}\theta.$$

(a) Prove that

$$D_N(\theta) = \frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}.$$

(b) Prove that there is $M > 0$ such that:

$$\left|\frac{1}{\sin\theta} - \frac{1}{\theta}\right| \leq M,$$

for $\theta \in [0, \pi/2]$.

(c) Prove that, for $N \geq 2$ we have[2]

$$L_N = \frac{2}{\pi}\int_0^{\pi} |\sin\theta| \Big(\sum_{k=1}^{N-1} \frac{1}{\pi k+\theta} \Big) \,\mathrm{d}\theta + O(1).$$

(d) Prove that, for $N \geq 2$:

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + O(1).$$

(e) Conclude that

$$\lim_{N\to\infty} \frac{L_N}{\log N} = \frac{4}{\pi^2}.$$

This implies that there is an universal constant $c > 0$ such that

$$\int_{-\pi}^{\pi} |D_N(\theta)| \,\mathrm{d}\theta \geq c\log N.$$

(31) Prove that there is $C > 0$ such that for any $N \geq 1$ and $[a, b] \subset [-\pi, \pi]$ we have

$$\left| \int_a^b D_N(\theta)\,\mathrm{d}\theta \right| \leq C.$$

(32) Let f be a $2\pi-$periodic and integrable function. Then

$$\lim_{n\to\infty} \widehat{f}(n) = 0.$$

It is known as Riemann-Lebesgue Lemma. (Hint: Use the approximation lemmas mentioned in class.)

(33)* Suppose that $f : [-\pi, \pi] \to \mathbb{C}$ such that $f(-\pi) = f(\pi)$, and $f \in C^1([-\pi, \pi])$. Let $S_N(f)$ the partial sums of the Fourier series of f.

(a) Shows that

$$S_N(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-y) - f(x))D_N(y)\,\mathrm{d}y,$$

where D_N is the Dirichlet kernel.

(b) Using Riemann-Lebesgue Lemma, conclude that for $x \in]-\pi, \pi[$,

$$S_N(f)(x) \to f(x),$$

as $N \to \infty$

(34) Prove that if a series of complex numbers $\sum c_n$ converges to $s \in \mathbb{C}$, then $\sum c_n$ is Cèsaro summable to s.

(35) Let $f : [0, 2\pi] \to \mathbb{C}$ be a continuous function such that $f(0) = f(2\pi)$. Determine the limit

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)(\cos(nx + n^3))^2 \,\mathrm{d}x.$$

[2] For a function f, the notation $f = O(1)$ means that $|f(x)| \leq M$ for some $M > 0$.

(36)* Let $N \geq 1$ be a natural number and let D_N be the Dirichlet kernel. Define

$$m(N) = \min_{x\in\mathbb{R}} D_N(x)$$

Show that $m(1) = -1$ and $m(2) = -5/4$. The purpose of the following items is to show that

$$\lim_{N\to\infty} \frac{m(N)}{N} = 2c_0, \tag{0.1}$$

where

$$c_0 = \min_{x\in\mathbb{R}} \frac{\sin x}{x} = -0.21723... \tag{0.2}$$

(a) Using the mean value theorem prove that

$$\left| \sum_{k=1}^{N} \cos(xk)\frac{1}{N} - \int_0^1 \cos(xNt)\,\mathrm{d}t \right| \leq \frac{x}{2},$$

for all $x \geq 0$.

(b) Deduce that, for $N \geq 1$ and $x > 0$,

$$\frac{1}{N} + \frac{2\sin(Nx)}{Nx} - x \leq \frac{D_N(x)}{N} \leq \frac{1}{N} + \frac{2\sin(Nx)}{Nx} + x.$$

(c) Prove that

$$m(N) = \min_{x\in[0,\pi]} D_N(x).$$

(d) Using the definition (0.2), deduce that there are constants $K_1, K_2 > 0$ such that

$$2c_0 - \frac{K_1}{N} \leq \frac{m(N)}{N} \leq 2c_0 + \frac{K_2}{N},$$

for all $N \geq 1$. Conclude (0.1).

"Blessed is the one who finds wisdom,
and the one who gets understanding,
for the gain from her is better than
gain from silver."

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