Fourier analysis - NTNU 2022
Instructor: Andrés Chirre

PROBLEM SET 3

(37) Show that the examples of inner product spaces, $\mathbb{R}^d$, and $\mathbb{C}^d$ are Hilbert spaces.

(38) Prove that the vector space $\ell^2(\mathbb{Z})$ is a Hilbert space.

(39) Construct a sequence of integrable functions $\{f_k\}_{k\geq 1}$ defined on $[0, 2\pi]$ such that

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |f_k(\theta)|^2 \,\mathrm{d}\theta = 0,$$

but $\lim_{k\to\infty}$ fails to exist for any $\theta \in [0, 2\pi]$.

(40) We recall that the vector space $\mathcal{R}$ of integrable functions, with its inner product and norm

$$\|f\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x\right)^{1/2}.$$

(a) Shpw that there exist a non-countable family of non-zero integrable functions f for which $\|f\| = 0$.
(b) Show that if $f \in \mathcal{R}$ with $\|f\| = 0$, then $f(x) = 0$ whenever f is continuous at x.
(c) Conversely, show that if $f \in \mathcal{R}$ vanishes at all of its points of continuity, then $\|f\| = 0$.

(41) Use the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(\theta) = |\theta|$, to find the value of the sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(42) Use the 2π-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$ to compute the values of

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6}.$$

(43) Let $\alpha \notin \mathbb{Z}$. Consider the function $g : [0, 2\pi] \to \mathbb{C}$ defined by

$$g(x) = \frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$$

(a) Compute the Fourier coefficients of g.
(b) Prove the identity

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

(44) A classical result in the theory of integration states that: let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integral functions such that $f(x) = g(x)$ outside of a set of measure zero. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Prove the above result, assuming that U is a finite set.

(45) Riemann-Lebesgue Lemma V3: Let $f : [0, 2\pi] \to \mathbb{C}$ be an integrable function, such that it does not necessarily satisfies $f(0) = f(2\pi)$.

(a) Prove that

$$\lim_{n\to\infty} \widehat{f}(n) = 0.$$

(b) Consider the subinterval $[a, b] \subset [0, 2\pi]$. Prove that

$$\lim_{n\to\infty} \int_a^b f(x)\cos(nx)\mathrm{d}x = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_a^b f(x)\sin(nx)\mathrm{d}x = 0.$$

In particular, prove that

$$\lim_{n\to\infty} \int_a^b f(x)\sin((n + 1/2)x)\mathrm{d}x = 0.$$

(46) Does there exist an integrable function on the circle $f : [-\pi, \pi] \to \mathbb{C}$ such that

$$\widehat{f}(n) = \frac{2}{\sqrt{1 + |n|}},$$

for all $n \in \mathbb{Z}$?

(47) Let $c \in \mathbb{R}$, and suppose that there is a continuous function on the circle $f : [0, 2\pi] \to \mathbb{C}$ such that

$$\widehat{f}(n) = \frac{c}{|n|},$$

for all $n \neq 0$. Prove that there is $x_0 \in [0, 2\pi]$ such that $|f(x_0)| \geq \frac{c\pi}{\sqrt{3}}$.

(48) Let $f : [-\pi, \pi] \to \mathbb{C}$ be an integrable function on the circle such that f is an odd function and

$$\widehat{f}(n) = \frac{1}{|n|},$$

for all $n \neq 0$. Prove that

$$\int_{-\pi}^{\pi} |f(x)|\,\mathrm{d}x \leq \frac{2\sqrt{3}\pi^2}{3}.$$

(Hint: Use Cauchy-Schwarz).

(49) Does there exist a 2π-periodic integrable function f such that $0 \leq f(x) \leq \pi$ for all $x \in [-\pi, \pi]$, and

$$\widehat{f}(n) = \frac{1}{\sqrt{1 + n^2}},$$

for all $n \in \mathbb{Z}$? (You can assume the fact that $\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi \frac{e^{2\pi}+1}{(e^{2\pi}-1)}$.)

(50) Dini's Test: Let $f : \mathbb{R} \to \mathbb{C}$ be a $2\pi-$periodic and integrable function.

(a) Let D_N be the Dirichlet kernel, and $S_N(f)$ the partial sums of the Fourier series of f. Show that, for $\theta \in \mathbb{R}$,

$$S_N(f)(\theta) - f(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big(f(\theta + u) + f(\theta - u) - 2f(\theta)\big) D_N(u)\,\mathrm{d}u.$$

(b) Let $\theta \in \mathbb{R}$, and suppose that for some $0 < \eta \leq \pi$, we have

$$\lim_{N\to\infty} \frac{1}{4\pi} \int_{-\eta}^{\eta} \big(f(\theta + u) + f(\theta - u) - 2f(\theta)\big) D_N(u)\,\mathrm{d}u = 0,$$

Prove that $S_N(f)(\theta) \to f(\theta)$, as $N \to \infty$. (Hint: Use the Problem (35.b)).

(c) Prove Dini's Test: Let $\theta \in \mathbb{R}$, and assume that there is $\delta > 0$ such that the function

$$g(u) = \left| \frac{f(\theta + u) + f(\theta - u) - 2f(\theta)}{u} \right|$$

is an integrable function on the interval $[-\delta, \delta] \subset [-\pi, \pi]$. Prove that $S_N(f)(\theta) \to f(\theta)$, as $N \to \infty$.

(51) Let $f : \mathbb{R} \to \mathbb{C}$ be a $2\pi-$periodic function, integrable on $[-\pi, \pi]$. Suppose that f satisfies a Hölder condition of order α, for some $0 < \alpha \leq 1$, i.e., for some $C > 0$ we have

$$|f(x+h) - f(x)| \leq C|h|^{\alpha},$$

for all $x, h \in \mathbb{R}$. Prove that

$$\widehat{f}(n) = O\left(\frac{1}{|n|^{\alpha}}\right).$$

Prove that the above result cannot be improved by showing the following statements:

(a) Let $0 < \alpha < 1$. Prove that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}$$

is a $2\pi-$periodic function, integrable on $[-\pi, \pi]$ and satisfies the Hölder condition of order α.

(b) For the above function, show that $\widehat{f}(n) = n^{-\alpha}$ whenever $n = 2^k$.

(52) Let f be a $2\pi-$periodic function which satisfies a Lipschitz condition with constant K; that is,

$$|f(x) - f(y)| \leq K|x - y|,$$

for $x, y \in \mathbb{R}$. This is simply the Hölder condition with $\alpha = 1$. We want to prove that the Fourier series of f converges absolutely and uniformly, following the next outline:

(a) For every positive h we define $g_h(x) = f(x+h) - f(x-h)$. Prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |g_h(x)|^2 \, \mathrm{d}x = \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2 |\widehat{f}(n)|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty} |\sin(nh)|^2 \, |\widehat{f}(n)|^2 \leq K^2 h^2.$$

(b) Let p be a positive integer. By choosing $h = \pi/2^{p+1}$, show that

$$\sum_{2^{p-1} < |n| \leq 2^p} \left|\widehat{f}(n)\right|^2 \leq \frac{K^2 \pi^2}{2^{2p+1}}.$$

(c) Estimate $\sum_{2^{p-1}<|n|\leq 2^p} \left|\widehat{f}(n)\right|$, and conclude that the Fourier series of f converges absolutely, hence uniformly. (Hint: Use the Cauchy-Schwarz inequality to estimate the sum.)

(d) In fact, modify the argument slightly to prove Bernstein's theorem: If f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series converges absolutely.

(53) Prove or disprove:

(a) For any enumeration of the rational numbers $\{\xi_n\}_{n\geq 1}$ in $[0,1)$ we have that $\{\xi_n\}_{n\geq 1}$ is equidistributed.

(b) There is an enumeration of the rational numbers $\{\xi_n\}_{n\geq 1}$ in $[0,1)$ such that $\{\xi_n\}_{n\geq 1}$ is equidistributed.

(54) Let γ be an irrational number, and let $f : [0,1] \to \mathbb{C}$ be a periodic Riemann integrable function of period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) = \int_0^1 f(x)\,\mathrm{d}x.$$

(55) Weyl's criterion: Let $\{\xi_n\}_{n\geq 1}$ be a sequence of real numbers in $[0,1)$. The following propositions are equivalent:

(a) $\{\xi_n\}_{n\geq 1}$ is equidistributed.

(b) For all $k \in \mathbb{Z}$, $k \neq 0$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0.$$

(56) Let $\{\xi_n\}_{n\geq 1}$ be an equidistributed sequence of real numbers in $[0,1)$ and $\{\alpha_n\}_{n\geq 1}$ be a sequence such that $\alpha_n \to 0$. Then, the sequence $\{(\xi_n + \alpha_n)\}_{n\geq 1}$ is equidistributed. [1]

(57) Show that, for any $a \neq 0$ and $0 < \sigma < 1$, the sequence $\{(an^\sigma)\}_{n\geq 1}$ is equidistributed in $[0,1)$.
Hint: Prove that for any fixed $b \neq 0$ and $N \geq 1$ we have

$$\sum_{n=1}^{N} e^{2\pi i b n^\sigma} - \int_1^N e^{2\pi i b x^\sigma}\,\mathrm{d}x = O\left(\sum_{n=1}^{N} n^{-1+\sigma}\right).$$

(58) Suppose that f is a periodic function on $\mathbb{R}$ of period 1, and $\{\xi_n\}_{n\geq 1}$ is a sequence equidistributed in $[0,1)$. Prove that:

(a) If f is continuous and satisfies $\int_0^1 f(x)\,\mathrm{d}x = 0$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+\xi_n) = 0,$$

uniformly in x.

(b) If f is merely integrable on $[0,1]$ and satisfies $\int_0^1 f(x)\,\mathrm{d}x = 0$, then

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} f(x+\xi_n) \right|^2 \mathrm{d}x = 0.$$

"Give ear to the training of your father,
and do not give up the teaching of your mother."

Email address: `carlos.a.c.chavez@ntnu.no`

[1] The notation (ξ_n) means the fractional part of ξ_n.