Fourier analysis - NTNU 2022
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PROBLEM SET 5

(72) Find the constant $C > 0$ such that

$$\int_{-\infty}^{\infty} \frac{e^{-\pi x^2}}{1+x^2}\,\mathrm{d}x = C \int_{1}^{\infty} e^{-\pi x^2}\,\mathrm{d}x.$$

(73) Let $a, b > 0$. Use the Fourier transform of the function $e^{-2\pi\lambda|x|}$ to compute

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)(b^2+x^2)}\,\mathrm{d}x.$$

(74) Let $f(x) = (x^2+2x+2)e^{-\pi x^2}$. Compute the Fourier transform of f.

(75) Let $g : \mathbb{R} \to \mathbb{R}$ be the function such that

$$e^{\pi t^2} g(t) = \int_{-\infty}^{\infty} (x^2+2x+2)e^{-2\pi x(x-t)}\,dx,$$

for all $t \in \mathbb{R}$. Compute the Fourier transform of g.

(76) For each $m \geq 1$ a natural number, we want to obtain a expression for

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}},$$

(a) Apply the Poisson summation formula to obtain, for $t > 0$ that:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi t|n|}.$$

(b) Prove the following identity valid for $0 < t < 1$:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}.$$

(c) Use the fact that

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m},$$

to deduce the formula

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

The numbers B_{2m} are known as the Bernoulli numbers. Compute $\zeta(6)$ and $\zeta(8)$.

(77) The following facts have been given in class as exercises.

(a) Prove the fomula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, for all $s \in \mathbb{C}$.

(b) Let $\eta(s)$ the analytic function defined on $\mathrm{Re}\, s > 0$ as $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ Show that $\eta(1) = \ln 2$. Also, show that $\eta(s)$ has zeros at the points $s = \frac{2\pi k i}{\ln 2} + 1$, where $k \in \mathbb{Z} \setminus \{0\}$.

(c) Prove that $\theta(\lambda) = \sum_{n=1}^{\infty} e^{-\pi\lambda n^2}$ is a continuous function on $(0, \infty)$.
(d) Justify the step $\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi\lambda n^2} \lambda^{s/2-1}\, \mathrm{d}\lambda = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi\lambda n^2} \lambda^{s/2-1}\, \mathrm{d}\lambda$, for $\operatorname{Re} s > 1$.
(e) Prove that $g : \mathbb{C} \to \mathbb{C}$ is an entire function, where

$$g(s) = \int_1^{\infty} \theta(u)\big(u^{-1/2-s/2} + u^{s/2-1}\big)\, \mathrm{d}u.$$

(78) Prove that, for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 0$ we have[1]

$$\sum_{n=-\infty}^{\infty} e^{-sn^2} = \sqrt{\frac{\pi}{s}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/s}.$$

(79) In class we have proved the Heisenberg uncertainty principle for $f \in \mathcal{S}(\mathbb{R})$. Which conditions are sufficient for $f \in \mathcal{M}(\mathbb{R})$ to obtain the desired inequality?

(80) Let $a, b > 0$. Let $f \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = 1$. Suppose that

$$\int_{-a}^{a} x^2 |f(x)|^2\, \mathrm{d}x \geq \frac{1}{2} \int_{-\infty}^{\infty} x^2 |f(x)|^2\, \mathrm{d}x,$$

and

$$\int_{-b}^{b} \xi^2 |\widehat{f}(\xi)|^2\, \mathrm{d}\xi \geq \frac{1}{2} \int_{-\infty}^{\infty} \xi^2 |\widehat{f}(\xi)|^2\, \mathrm{d}\xi.$$

Prove that $ab \geq 1/8\pi$.

(81) Suppose that $f \in \mathcal{M}(\mathbb{R})$ such that f and $\widehat{f}$ have compact supports. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(82) Prove Poincare's inequality: for all $f \in C^2([-\frac{1}{2}, \frac{1}{2}])$ such that $f(-\frac{1}{2}) = f(\frac{1}{2})$ we have

$$\int_{-1/2}^{1/2} \left(f(x) - \int_{-1/2}^{1/2} f(t)\, \mathrm{d}t \right)^2 \mathrm{d}x \leq \frac{1}{4\pi^2} \int_{-1/2}^{1/2} (f'(x))^2 \mathrm{d}x.$$

For which functions does equality hold? (Hint: Use fourier series).

(83) Let $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f}(x) = 0$ for $|x| \geq 1$. Suppose that

$$f(x) \geq e^{-2\pi|x|}$$

for all $x \in \mathbb{R}$. Show that $\widehat{f}(0) \geq (e^{2\pi} + 1)/(e^{2\pi} - 1)$.

(84) Let $f \in \mathcal{M}(\mathbb{R})$ such that for all $\xi \in \mathbb{R}$ we have $|\widehat{f}(\xi)| \leq A e^{-2\pi a|\xi|}$, for some constants $a, A > 0$. Prove that $f(x)$ is the restriction to $\mathbb{R}$ of a function $f(z)$ holomorphic in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < b\}$ for any $0 < b < a$.

(85) Let $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f}(\xi) = O\big(e^{-2\pi a|\xi|}\big)$, for some constants $a > 0$. Suppose that $f(1/n) = 0$ for all $n \in \mathbb{N}$. Prove that $f \equiv 0$.

[1] The complex root for s is defined such that $\sqrt{1} = 1$.

(86) Prove the following version of Phragmén Lindelöf Theorem: Suppose that F is a holomorphic function in the sector $\mathcal{S} = \{z \in \mathbb{C} : -\pi/4 < \arg z < \pi/4\}$ that is continuous on the closure of $\mathcal{S}$. Assume $|F(z)| \leq 1$ on the boundary of $\mathcal{S}$, and that there are constants $C, c > 0$ such that for all z in $\mathcal{S}$ we have $|F(z)| \leq Ce^{c|z|}$. Then $|F(z)| \leq 1$ for all $z \in \mathcal{S}$.

(87) Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that $f \in \mathcal{M}(\mathbb{R})$ and $\operatorname{supp} \widehat{f} \subset [-\frac{1}{2}, \frac{1}{2}]$. Suppose that $f(n) = 0$ for all $n \in \mathbb{Z} \setminus \{-1, 1\}$ and $f(-1) + f(1) = 0$.

(a) Prove that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 0$.

(b) Prove that, if $f(1) = i$, then

$$f(x) = -\frac{2i\sin(\pi x)}{\pi(x+1)(x-1)}.$$

(c) Use the above function to prove that

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi(x^2-1)}\right)^2 \mathrm{d}x = \frac{1}{2}.$$

(88) Let $\mathcal{F}$ be the family of functions $f : \mathbb{R} \to \mathbb{R}$ such that satisfying the following properties: $f \in \mathcal{M}(\mathbb{R})$, $f(x) \geq 0$ for all $x \in \mathbb{R}$, $f(0) = 1$, and $\operatorname{supp} \widehat{f} \subset [-1, 1]$. Find

$$\inf_{f \in \mathcal{F}} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Moreover, show that there is a unique function $f \in \mathcal{F}$ such that the above infimum is attained.

(89) Let $f \in \mathcal{M}(\mathbb{R})$ such that $\widehat{f}(x) = 0$ for $|x| \geq 1$. Suppose that

$$f(x) \geq \frac{1}{1+x^2}.$$

for all $x \in \mathbb{R}$. Show that $\widehat{f}(0) \geq \pi(e^{2\pi}+1)/(e^{2\pi}-1)$. In fact, prove that there is a unique function satisfying the mentioned properties and $\widehat{f}(0) = \pi(e^{2\pi}+1)/(e^{2\pi}-1)$.

(90) Define the functions

$$m^{\pm}(z) = \frac{1}{1+z^2}\left(\frac{e^{2\pi}+e^{-2\pi}-2\cos(2\pi z)}{(e^{\pi} \mp e^{-\pi})^2}\right)$$

Prove that $m^{\pm}(z)$ are entire functions of exponential type 2π such that their Fourier transforms have compact supports in $[-1, 1]$. Also prove that

$$m^-(x) \leq \frac{1}{1+x^2} \leq m^+(x) \quad \text{for all } x \in \mathbb{R}.$$

"The heart of the man of good sense gets knowledge;
the ear of the wise is searching for knowledge."

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