Fourier analysis exercise 34 & 35

Markus

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Exercise 34)

If $a = b$ the statement is trivial so assume $[a, b]$ has non-empty interior. Let $U = \{x_1, \dots, x_n\}$ with without loss of generalization $x_1 < x_2 < \cdots < x_n$. Let $\varepsilon > 0$. Assuming $x_i \neq a, b$, we choose a partition

$$\Pi_\varepsilon = \left[a, x_1 - \frac{\varepsilon}{2n}\right] \cup \left[x_1 - \frac{\varepsilon}{2n}, x_1 + \frac{\varepsilon}{2n}\right] \cup \left[x_1 + \frac{\varepsilon}{2n}, x_2 - \frac{\varepsilon}{2n}\right] \cup \left[x_2 - \frac{\varepsilon}{2n}, x_2 + \frac{\varepsilon}{2n}\right] \cup \cdots \cup \left[x_n + \frac{\varepsilon}{2n}, b\right]$$

if $x_1 = a$ or $x_n = b$ we change the partition accordingly. We can without loss of generalization choose ε so small that no x_i is in two sets of the partition, and so that $a < x_1 - \frac{\varepsilon}{2n}$, $b > x_n + \frac{\epsilon}{2n}$. Since $f - g$ is defined on all of $[x_i - \frac{\varepsilon}{2n}, x_i + \frac{\varepsilon}{2n}]$ we can define the two following (finite) constants

$$B = \max_{1 \le i \le n} \sup_{x : |x - x_i| \le \frac{\varepsilon}{2n}} (f - g)(x)$$

$$N = \min_{1 \le i \le n} \inf_{x : |x - x_i| \le \frac{\varepsilon}{2n}} (f - g)(x)$$

We observe that

$$B = \max_{1 \le i \le n} \max\{0, (f - g)(x_i)\} \qquad N = \min_{1 \le i \le n} \min\{0, (f - g)(x_i)\}$$

and hence $B \ge 0$ and $N \le 0$, and furthermore N, B are indepedent of ε. Observe that on

$$\left[a, x_1 - \frac{\varepsilon}{2n}\right], \left[x_i + \frac{\varepsilon}{2n}, x_i - \frac{\varepsilon}{2n}\right], \left[x_n + \frac{\varepsilon}{2n}, b\right]$$

we have $f - g = 0$. Hence we have

$$L(f - g, \Pi_\varepsilon) = \sum_{i=1}^{n} \inf_{x : |x - x_i| \le \varepsilon/2n} (f - g)(x) \left(x_i + \frac{\varepsilon}{2n} - \left(x_i - \frac{\varepsilon}{2n}\right)\right) \ge N\varepsilon$$

$$U(f - g, \Pi_\varepsilon) = \sum_{i=1}^{n} \sup_{x : |x - x_i| \le \varepsilon/2n} (f - g)(x) \left(x_i + \frac{\varepsilon}{2n} - \left(x_i - \frac{\varepsilon}{2n}\right)\right) \le B\varepsilon$$

Since $f-g$ is Riemann-integrable (it is a difference of Riemann-integrable functions) we get

$$\int_a^b (f-g)\,\mathrm{d}x = \inf_{\text{partitions } \mathcal{P}} U(f-g,\mathcal{P}) \leq \inf_{\Pi_\varepsilon:\varepsilon>0} U(f-g,\Pi_\varepsilon) \leq \inf_{\Pi_\varepsilon:\varepsilon>0} B\varepsilon = 0$$

$$0 = \sup_{\Pi_\varepsilon:\varepsilon>0} N\varepsilon \leq \sup_{\Pi_\varepsilon:\varepsilon>0} L(f-g,\Pi_\varepsilon) \leq \sup_{\text{partitions } \mathcal{P}} L(f-g,\mathcal{P}) = \int_a^b (f-g)\,\mathrm{d}x$$

For the last supremum, remember $N \leq 0$. Hence we conclude that

$$\int_a^b (f-g)\mathrm{d}x = 0$$

and so

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$$

Problem 35)

a)

Define $g : [0,2\pi] \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x) & x \in [0,2\pi) \\ f(0) & x = 2\pi \end{cases}$$

Then g is integrable, so by Problem 34) it follows that

$$\int_0^{2\pi} g(x)e^{-inx}\,\mathrm{d}x = \int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x$$

Furthermore g is integrable on the circle so Riemann-Lebesgue lemma apply to g and we get

$$0 = \lim_{|n|\to\infty} \int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x$$

b)

Define $g : [0,2\pi] \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x) & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Then g satifies the assumptions of a) and we get from a) that

$$0 = \lim_{|n|\to\infty} \int_0^{2\pi} g(x)e^{-inx}\,\mathrm{d}x = \lim_{|n|\to\infty} \int_a^b f(x)e^{-inx}\,\mathrm{d}x$$

Now

$$\int_a^b f(x)\cos(nx)\,\mathrm{d}x = \int_a^b f(x)\left(\frac{e^{inx}+e^{-inx}}{2}\right)\,\mathrm{d}x = \frac{1}{2}\left(\int_a^b f(x)e^{inx} + \int_a^b f(x)e^{-inx}\right)$$

By the first part of b) both those integral tends to 0 in the limit. The case for sin is analogous - use the same decomposition and the same argument applies. For the final part we observe that

$$\sin\left(\left(N+\frac{1}{2}\right)x\right) = \sin(Nx)\cos(x/2) + \cos(Nx)\sin(x/2)$$

and then we apply the Riemann-Lebesgue lemma on $[a, b]$ which we have proved here in b) to the functions $f(x)\cos(x/2)$ and $f(x)\sin(x/2)$.