Fourier analysis - NTNU 2022
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SOLUTION PROBLEM 36

1. Minima of Dirichlet kernels

For $n \in \mathbb{Z}_{\geqslant 0}$ we consider the *Dirichlet kernel* D_n given by

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin\big((n+1/2)x\big)}{\sin(x/2)}. \tag{1.1}$$

Let us define the minimum

$$\mathfrak{m}(n) := \min_{\theta \in [0,2\pi]} \frac{\sin\big((2n+1)\theta\big)}{\sin\theta} = \min_{x\in\mathbb{R}} D_n(x)\,, \tag{1.2}$$

and the universal constant

$$c_0 := \min_{x\in\mathbb{R}} \frac{\sin x}{x} = -0.21723\ldots. \tag{1.3}$$

Proposition 1. *For each $n \in \mathbb{N}$, the following bounds hold*

$$2c_0 - \frac{(2\pi-1)}{n} \;\leqslant\; \frac{\mathfrak{m}(n)}{n} \;\leqslant\; 2c_0 + \frac{5.4935}{n}.$$

Proof. We rewrite (1.1) as

$$D_n(n) = 1 + 2n\sum_{k=1}^{n} \cos\left(xn\frac{k}{n}\right)\frac{1}{n}.$$

Using the mean value theorem we get, for $x \geqslant 0$,

$$\begin{aligned}
\left|\sum_{k=1}^{n} \cos\left(xn\frac{k}{n}\right)\frac{1}{n} - \int_0^1 \cos(xnt)\,\mathrm{d}t\right| &= \left|\sum_{k=1}^{n} \cos\left(xn\frac{k}{n}\right)\frac{1}{n} - \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} \cos(xnt)\,\mathrm{d}t\right| \\
&\leqslant \sum_{k=1}^{n}\left|\int_{(k-1)/n}^{k/n}\left(\cos\left(xn\frac{k}{n}\right) - \cos(xnt)\right)\mathrm{d}t\right| \\
&\leqslant \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} xn\left(\frac{k}{n} - t\right)\mathrm{d}t \\
&= \frac{x}{2}.
\end{aligned}$$

Therefore,

$$\frac{1}{n} + \frac{2\sin(nx)}{nx} - x \leqslant \frac{D_n(x)}{n} \leqslant \frac{1}{n} + \frac{2\sin(nx)}{nx} + x. \tag{1.4}$$

Let $x_1 = 4.49340\ldots$ be the unique real positive number such that

$$c_0 = \min_{x\in\mathbb{R}} \frac{\sin x}{x} = \frac{\sin x_1}{x_1} = -0.21723\ldots.$$

Plugging $x_n = x_1/n$ in (1.4) we obtain

$$\frac{\mathfrak{m}(n)}{n} \leqslant \frac{D_n(x_n)}{n} \leqslant \frac{1}{n} + \frac{2\sin(nx_n)}{nx_n} + x_n = \frac{1}{n} + \frac{2\sin(x_1)}{x_1} + \frac{x_1}{n} \leqslant 2c_0 + \frac{5.4935}{n}.$$

On the other hand, using the fact that $D_n(x)$ is an even periodic function with period 2π, it follows that $\mathfrak{m}(n) = \min_{x\in[0,\pi]} D_n(x)$. Let $\xi \in [0,\pi]$ be a real number where such minimum is attained. If $2\pi/(2n+1) \leqslant \xi \leqslant$

$4\pi/(2n+1)$, using (1.4) we get

$$\frac{\mathfrak{m}(n)}{n} = \frac{D_n(\xi)}{n} \geqslant \frac{1}{n} + 2c_0 - \frac{4\pi}{2n+1} > 2c_0 - \frac{(2\pi-1)}{n}.$$

If $6\pi/(2n+1) \leqslant \xi \leqslant \pi$, using the fact that $\sin t \geqslant 2t/\pi$ for $t \in \left[0, \frac{\pi}{2}\right]$ we have

$$\frac{\mathfrak{m}(n)}{n} = \frac{D_n(\xi)}{n} = \frac{\sin((n+1/2)\xi)}{n\sin(\xi/2)} \geqslant \frac{-1}{n\sin(\xi/2)} \geqslant -\frac{\pi}{n\xi} \geqslant -\frac{2n+1}{6n} > 2c_0 - \frac{(2\pi-1)}{n}.$$

Finally, in the cases $0 \leqslant \xi < 2\pi/(2n+1)$ or $4\pi/(2n+1) < \xi < 6\pi/(2n+1)$, it is clear that $D_n(\xi) \geqslant 0$, and such points will not be points where the global minimum is attained. This concludes the proof. ☐

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