38

(38) Let f be a 2π−periodic function which satisfies a Lipschitz condition with constant K; that is,

$$|f(x) - f(y)| \leq K|x - y|,$$

for $x, y \in \mathbb{R}$. This is simply the Hölder condition with $\alpha = 1$. We want to prove that the Fourier series of f converges absolutely and uniformly, following the next outline:

(a) For every positive h we define $g_h(x) = f(x + h) - f(x - h)$. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2 |\widehat{f}(n)|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty} |\sin(nh)|^2 \, |\widehat{f}(n)|^2 \leq K^2 h^2.$$

(b) Let p be a positive integer. By choosing $h = \pi/2^{p+1}$, show that

$$\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

(c) Estimate $\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|$, and conclude that the Fourier series of f converges absolutely, hence uniformly. (Hint: Use the Cauchy-Schwarz inequality to estimate the sum.)

(d) In fact, modify the argument slightly to prove Bernstein's theorem: If f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series converges absolutely.

Let f be 2π-periodic s.t. $\exists K > 0$:

$$|f(x) - f(y)| \leq K|x - y| \quad \text{for } x, y \in \mathbb{R}.$$

[This is the Hölder condition with $\alpha = 1$. Aka: Lipschitz.]

↳ So we know $\hat{f}(n) = O\left(\frac{1}{|n|}\right)$ from problem 37. But not relevant.

(a) For all positive h, let $g_h(x) := f(x+h) - f(x-h)$.

Note that since f is Lipschitz, it is also continuous.

So g_h is continuous in the variable x.

First we calculate $\hat{g}_h(n) = \widehat{f(x+h)}(n) - \widehat{f(x-h)}(n)$

Then we use Parseval on g_h:

By problem 1, we can shift the integrand, since it is periodic.

$$\widehat{f(x+h)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h) e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-in(x-h)} dx$$

$$= e^{inh} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = e^{inh} \hat{f}(n).$$

Same for $\widehat{f(x-h)}(n) = e^{-inh}\hat{f}(n)$.

So: $$\begin{aligned}\hat{g}_h(n) &= \widehat{f(x+h)}(n) - \widehat{f(x-h)}(n) \\ &= e^{inh}\hat{f}(n) - e^{-inh}\hat{f}(n) \\ &= \left(e^{inh} - e^{-inh}\right)\hat{f}(n) \\ \hat{g}_h(n) &\underset{\uparrow}{=} 2i\sin(nh)\,\hat{f}(n).\end{aligned}$$

Euler formula.

Note that g_h is also 2π-periodic:
$$\begin{aligned}g_h(x+2\pi) &= f(x+2\pi+h) - f(x+2\pi-h) \\ &\underset{\uparrow}{=} f(x+h) - f(x-h) = g_h(x).\end{aligned}$$

f is periodic.

So we may use Parseval's Identity $\|g_h\|^2 = \sum_{n\in\mathbb{Z}} |\hat{g}_h(n)|^2$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2\,dx = \sum_{n\in\mathbb{Z}} 4\,|\sin(nh)|^2\,|\hat{f}(n)|^2. \qquad (*)$$

But $|g_h(x)| = |f(x+h) - f(x-h)| \underset{\uparrow}{\leq} K\,|x+h-(x-h)| = K\cdot 2h$,

So $|g_h(x)|^2 \leq 4K^2h^2$.

f is Lipschitz.

And $(*)$ becomes:
$$\begin{aligned}\sum_{n\in\mathbb{Z}} 4\,|\sin(nh)|^2\,|\hat{f}(n)|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2\,dx \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} 4K^2h^2\,dx \\ &= \frac{1}{2\pi}\,4K^2h^2\cdot 2\pi = 4K^2h^2.\end{aligned}$$

$$\Rightarrow \quad \sum_{n\in\mathbb{Z}} |\sin(nh)|^2 |\hat{f}(n)|^2 \le K^2h^2. \qquad (*)$$

(b) Let p be a positive integer and let $h = \frac{\pi}{2^{p+1}}$.

WTS: $$\sum_{2^{p-1} < |n| \le 2^p} |\hat{f}(n)|^2 \le \frac{K^2\pi^2}{2^{2p+1}}$$

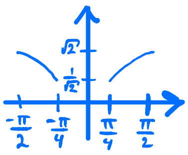
For $h = \frac{\pi}{2^{p+1}}$ we get by $(*)$:

$$\sum_{n=-\infty}^{+\infty} \left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2 |\hat{f}(n)|^2 \le \frac{K^2\pi^2}{2^{2p+2}},$$

so: $$\sum_{2^{p-1} < |n| \le 2^p} \left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2 |\hat{f}(n)|^2 \le \frac{K^2\pi^2}{2^{2p+2}}.$$

Note that on $2^{p-1} < |n| \le 2^p \Rightarrow \frac{\pi}{4} < \frac{\pi|n|}{2^{p+1}} \le \frac{\pi}{2}$,
and $\left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2$ is an increasing function for increasing $|n|$ on this set.

$|\sin(x)|^2$:



So: $$\sum_{2^{p-1} < |n| \le 2^p} \left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2 |\hat{f}(n)|^2 \geq \sum_{2^{p-1} < |n| \le 2^p} \left|\sin\left(\frac{\pi}{4}\right)\right|^2 |\hat{f}(n)|^2.$$

$$= \sum_{2^{p-1} < |n| < 2^p} \left(\frac{1}{\sqrt{2}}\right)^2 |\hat{f}(n)|^2.$$

$$\text{So: } \frac{1}{2} \sum_{2^{p-1} < |n| < 2^p} |\hat{f}(n)|^2 \leq \sum_{2^{p-1} < |n| < 2^p} \left|\sin\left(\pi \frac{n}{2^{p+1}}\right)\right|^2 |\hat{f}(n)|^2 \leq \frac{K^2 \pi^2}{2^{2p+2}}$$

$$\Rightarrow \sum_{2^{p-1} < |n| < 2^p} |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

c)

(c) Estimate $\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|$, and conclude that the Fourier series of f converges absolutely, hence uniformly. (Hint: Use the Cauchy-Schwarz inequality to estimate the sum.)

$$\left(\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)| \right)^2 = \left(\sum_{2^{p-1} < |n| \leq 2^p} 1 \cdot |\hat{f}(n)| \right)^2$$

C-S ineq.

$$\leq \left(\sum_{2^{p-1} < |n| \leq 2^p} 1^2 \right) \left(\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \right)$$

$$\leq 2^p \cdot \frac{K^2\pi^2}{2^{2p+1}}$$

$$= 2^p \cdot \frac{K^2\pi^2}{2^{p+1} \cdot 2^p}$$

$$= \frac{K^2\pi^2}{2^{p+1}}.$$

$$\Rightarrow \sum_{2^{p-1} < |n| < 2^p} |\hat{f}(n)| < \frac{K\pi}{2^{p/2 + 1/2}}.$$

So: $$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{p=1}^{\infty} \sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|$$

$$\le |\hat{f}(0)| + \sum_{p=1}^{\infty} \frac{K\pi}{2^{p/2+1/2}}$$

$$< |\hat{f}(0)| + \frac{K\pi}{\sqrt{2}} \sum_{p=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^p$$

$$= |\hat{f}(0)| + \frac{K\pi}{\sqrt{2}} \frac{1}{1-\frac{1}{\sqrt{2}}}$$

$$= |\hat{f}(0)| + K\pi \frac{1}{\sqrt{2}-1} < \infty.$$

$\Rightarrow \sum_{n\in\mathbb{Z}} \hat{f}(n)$ converges absolutely. $(*)$

But if the sum of the Fourier coefficients, $\sum_{n\in\mathbb{Z}} \hat{f}(n)$, converges absolutely, then we may apply the Weierstrass M-test to the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$: [Ref: Problem 10 Exercise 1]

Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence of functions defined on $\mathbb{R}$, with

$$f_n(x) := \hat{f}(n) e^{inx}.$$

Then $|f_n(x)| \leq M_n := |\hat{f}(n)|$,

and $\sum_{n\in\mathbb{Z}} M_n = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$ converges by $(*)$.

Therefore $\sum_{n\in\mathbb{Z}} f_n(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$ converges absolutely & uniformly.

Alternatively, since f is Lipschitz, then in particular it is continuous.
Then by Thm 1 in Class 14:

Theorem 1: f cont. on the circle and $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then

$$S_N(f)(\theta) \xrightarrow[n\to\infty]{} f(\theta) \quad \forall\, \theta \in \mathbb{R}.$$

"uniformly"

d)

(d) In fact, modify the argument slightly to prove Bernstein's theorem: If f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series converges absolutely.

Let now f be 2π-periodic and satisfying a Hölder condition of order $\alpha > 1/2$:

$$|f(x) - f(y)| \leq K|x-y|^{\alpha} \quad \text{for all } x,y \in \mathbb{R}.$$

WTS: $\sum_{n\in\mathbb{Z}} \hat{f}(n)$ converges absolutely.

We proceed as before: $g_h(x) := f(x+h) - f(x-h)$.

$$\Longrightarrow \quad \hat{g}_h(n) = 2i\sin(nh)\hat{f}(n).$$

And Parseval gives:

$$\sum_{n=-\infty}^{+\infty} 4|\sin(nh)|^2 |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2 dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^2 dx$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} K^2\left((2h)^{\alpha}\right)^2 dx$$

$$= 2^{2\alpha} K^2 h^{2\alpha}.$$

$$\Longrightarrow \sum_{n=-\infty}^{+\infty} |\sin(nh)|^2 |\hat{f}(n)|^2 \leq \frac{2^{2\alpha}}{4} K^2 h^{2\alpha} = 2^{2(\alpha-1)} K^2 h^{2\alpha}.$$

Let now p be a positive integer, and pick $h = \frac{\pi}{2^{p+1}}$.

Then, for $2^{p-1} < |n| \leq 2^p \Rightarrow \frac{\pi}{4} < \pi\frac{|n|}{2^{p+1}} < \frac{\pi}{2}$.

$$\Rightarrow \left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2 \geq \left|\sin\left(\frac{\pi}{4}\right)\right|^2 = \frac{1}{2}.$$

So:
$$\frac{1}{2}\cdot\sum_{2^{p-1}<|n|\leq 2^p}|\hat{f}(n)|^2 \leq \sum_{2^{p-1}<|n|\leq 2^p}\left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2|\hat{f}(n)|^2$$
$$\leq \sum_{n=-\infty}^{\infty}\left|\sin\left(\pi\frac{n}{2^{p+1}}\right)\right|^2|\hat{f}(n)|^2$$
$$\leq 2^{2(\alpha-1)}K^2h^{2\alpha} = 2^{2(\alpha-1)}\cdot K^2\left(\frac{\pi}{2^{p+1}}\right)^{2\alpha}$$

$$\Rightarrow \sum_{2^{p-1}<|n|\leq 2^p}|\hat{f}(n)|^2 \leq 2^{2\alpha-1}\cdot K^2\cdot\frac{\pi^{2\alpha}}{2^{2\alpha(p+1)}} = \frac{K^2\cdot\pi^{2\alpha}}{2^{2\alpha p+1}}.$$

We now want to estimate $\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|$ using Cauchy-Schwarz:

$$\left(\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)| \right)^2 = \left(\sum_{2^{p-1} < |n| \leq 2^p} 1 \cdot |\hat{f}(n)| \right)^2$$

$$\leq \left(\sum_{2^{p-1} < |n| \leq 2^p} 1^2 \right) \left(\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \right)$$

$$\leq 2^p \cdot \frac{K^2 \cdot \pi^{2\alpha}}{2^{2\alpha p + 1}}$$

$$\leq \frac{K^2 \pi^{2\alpha}}{2^{(2\alpha - 1)p + 1}}$$

$$\Rightarrow \sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)| \leq \frac{K \pi^\alpha}{2^{\frac{2\alpha - 1}{2} \cdot p} \cdot 2^{1/2}} = \frac{K \pi^\alpha}{2^{1/2}} \cdot 2^{(\frac{1}{2} - \alpha)p}$$

And finally:

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{p=1}^{\infty} \sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|$$

$$\leq |\hat{f}(0)| + \sum_{p=1}^{\infty} \frac{K \pi^\alpha}{2^{1/2}} \cdot 2^{(\frac{1}{2} - \alpha)p}$$

$$= |\hat{f}(0)| + K \pi^\alpha \cdot 2^{-1/2} \underbrace{\sum_{p=1}^{\infty} \left(2^{\frac{1}{2} - \alpha} \right)^p}_{\text{This geometric series converges whenever } \alpha > 1/2.} < \infty.$$

So $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|$ converges absolutely.

And as in part c) we apply the M-test to conclude that $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$ converges absolutely and uniformly.

[We may also apply Thm 1, since f Hölder $\Rightarrow$ f cont.]