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(61) Let $a, b > 0$. Use the Fourier transform of the function $e^{-2\pi\lambda|x|}$ to compute

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Recall: $P_\lambda(x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2+x^2} \Rightarrow \widehat{P}_\lambda(x) = e^{-2\pi\lambda|x|}$

And $\widehat{\widehat{P}}_\lambda(x) = P_\lambda(-x)$

Two times Fourier transform is reflection.

$$\int_{-\infty}^{\infty} \frac{1}{a^2+x^2}\cdot\frac{1}{b^2+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\pi}{a}\frac{1}{\pi}\frac{a}{a^2+x^2}\cdot\frac{\pi}{b}\frac{1}{\pi}\frac{b}{b^2+x^2}\,dx$$

$$= \frac{\pi^2}{ab}\int_{-\infty}^{\infty} P_a(x)\, P_b(x)\,dx$$

$P_a(x)$ is even.

$$= \frac{\pi^2}{ab}\int_{-\infty}^{\infty} P_a(-x)\, P_b(x)\,dx$$

$P_\lambda, \widehat{P}_\lambda \in M(\mathbb{R})$: Reflection Formula

$$= \frac{\pi^2}{ab}\int_{-\infty}^{\infty} \widehat{\widehat{P}}_a(x)\, P_b(x)\,dx$$

Multiplication formula

$$= \frac{\pi^2}{ab}\int_{-\infty}^{\infty} \widehat{P}_a(x)\, \widehat{P}_b(x)\,dx$$

$$= \frac{\pi^2}{ab} \int_{-\infty}^{\infty} e^{-2\pi a|x|} \cdot e^{-2\pi b|x|} \, dx$$

$$= \frac{\pi^2}{ab} \int_{-\infty}^{\infty} e^{-2\pi(a+b)|x|} \, dx$$

$$= \frac{\pi^2}{ab} \, 2 \int_{0}^{\infty} e^{-2\pi(a+b)x} \, dx$$

$$= \frac{\pi^2}{ab} \, 2 \left[\frac{e^{-2\pi(a+b)x}}{-2\pi(a+b)} \right]_{x=0}^{x=\infty \to 0}$$

$$= \frac{\pi^{\cancel{2}}}{ab} \, \cancel{2} \, \frac{1}{\cancel{2}\cancel{\pi}(a+b)}$$

$$= \frac{\pi}{ab(a+b)} .$$