Fourier analysis exercise 67

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Exercise 67)

I will present two solutions: one short and one which is a bit more elementary.

Solution 1)

By Paley-Wiener there is an entire function g such that $g|_{\mathbb{R}} = f$ (since $\operatorname{supp}(\widehat{f})$ has compact support), but since f has compact support g is 0 on a non-empty interval of $\mathbb{R}$, and hence the identity theorem implies $g = 0$.

Solution 2)

Say $\operatorname{supp}(f) \subset [-M, M]$ such that $f(-M) = f(M) = 0$. Since the support is compact, we can also choose M so big that there is an interval $(a, b) \in [-M, M]$ such that $f(x) = 0$ on (a, b). Then

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x = \int_{-M}^{M} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x$$

Since $f(-M) = f(M) = 0$ and $f(x) = 0$ for all $|x| \geq M$ we can look at f as a $2M$-periodic function - that is we will now change f with its periodization on $[-M, M]$, and just call this f. The associated Fourier coefficient is

$$\frac{1}{2M}\int_{-M}^{M} f(x)e^{\frac{-2\pi i x n}{2M}}\,\mathrm{d}x$$

Since $\widehat{f}$ has compact support, there is some K such that whenever $|n| > K$, $\widehat{f}(n) = 0$. Thus the Fourier series of f is finite, that is: it is a (dilated) trigonometric polynomial. Also by this we get $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$ since there are only finitely many non-zero terms in this sum. We have proven earlier that under the assumption that f is continuous and that the Fourier series converge absolutely then the Fourier series (of f) converge to f uniformly. Hence f is equal to a (dilated) trigonometric polynomial

$$f(x) = \sum_{|n|\leq K} \widehat{f}(n)e^{\frac{2\pi i n x}{2M}}$$

This extends to an entire function the obvious way:

$$g(z) = \sum_{|n| \leq K} \widehat{f}(n) e^{\frac{2\pi i n z}{2M}}$$

But $g|_{(a,b)} = f|_{(a,b)} = 0$, and (a, b) contains an accumulation point, hence the identity theorem gives that $g \equiv 0$. Hence $g|_{\mathbb{R}} = f(x) = 0$. Since we changed the original function f with its periodization on $[-M, M]$, this implies the original function f is 0 on $[-M, M]$, but it was 0 outside this set so $f(x) = 0$ for all $x \in \mathbb{R}$.