NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4170 Fourieranalyse**

Academic contact during examination: Andrés Chirre

Phone: 957 46 324

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Problem 1 Let f be the 2π-periodic function defined by $f(x) = x(x^2 - \pi^2)$ in $[-\pi, \pi]$.

a) Compute the n-Fourier coefficient of f, for all $n \in \mathbb{Z}$.

b) Prove that the Fourier series of f converges uniformly to f.

c) Prove that $\zeta(6) = \dfrac{\pi^6}{945}$, where ζ denotes the Riemann zeta-function.

SOLUTION:

a) $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)\,dx = 0$, because f is an odd-function.

Let $n \in \mathbb{Z}$, $n \neq 0$. Using integration by parts -times, we get:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)\left(\frac{e^{-inx}}{-in}\right)'dx = \frac{1}{2\pi}\left[-\int_{-\pi}^{\pi}(3x^2-\pi^2)\frac{e^{-inx}}{(-in)}\,dx\right]$$

$$= \frac{1}{2\pi i n}\int_{-\pi}^{\pi}(3x^2-\pi^2)\left(\frac{e^{-inx}}{-in}\right)'dx = \frac{1}{2\pi i n}\left[\frac{2\pi^2 e^{-in\pi}}{(-in)} - \frac{2\pi^2 e^{in\pi}}{(-in)} - \int_{-\pi}^{\pi} 6x\cdot\frac{e^{-inx}}{(-in)}\,dx\right]$$

$e^{in\pi} = (-1)^n$

$$= \frac{6}{2\pi(in)^2}\int_{-\pi}^{\pi} x\left(\frac{e^{-inx}}{-in}\right)'dx = \frac{-3}{\pi n^2}\left\{\frac{\pi e^{-in\pi}}{-in} - \frac{(-\pi)e^{in\pi}}{(-in)} - \int_{-\pi}^{\pi}\frac{e^{-inx}}{-in}\,dx\right\}$$

$$= \frac{-3}{\pi n^2}\left\{\frac{\pi(-1)^n}{-in} - \frac{\pi(-1)^n}{in} + \frac{1}{in}\left\{\frac{e^{-in\pi} - e^{in\pi}}{-in}\right\}\right\} = \frac{-3}{\pi n^2}\cdot\frac{-2\pi(-1)^n}{in} = \frac{6(-1)^n}{in^3}$$

Therefore
$$\hat{f}(n) = \begin{cases} 0, & n = 0 \\ \dfrac{6(-1)^n}{in^3}, & n \neq 0 \end{cases}$$

b) Since f is continuous and $\sum_{n\in\mathbb{Z}}|\hat{f}(n)| = \sum_{n\neq 0}\frac{6}{n^3} < \infty$, we obtain that

$$\sum_{|n|\le N}\frac{6(-1)^n}{in^3}e^{inx} \xrightarrow[N\to\infty]{} f(x) \text{ uniformly in } x \in [-\pi,\pi].$$

c) Using Parseval's identity:
$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{36}{n^6} = \sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2(x^2-\pi^2)^2\,dx = \frac{8\pi^6}{105}$$

$$\Rightarrow \zeta(6) = \sum_{n\geq 1}\frac{1}{n^6} = \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n^6} = \frac{1}{2}\cdot\frac{8\pi^6}{36\cdot 105} = \frac{\pi^6}{945}$$

Problem 2 Does there exist a 2π-periodic integrable function f such that $0 \le f(x) \le \pi$ for all $x \in [-\pi, \pi]$, and

$$\widehat{f}(n) = \frac{(-1)^n}{\sqrt{1+n^2}},$$

for all $n \in \mathbb{Z}$?

SOLUTION: Suppose that such function exists. Using Parseval identity we have

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx \quad \dots (*)$$

Note that, using Poisson summation formula we get

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi \sum_{n\in\mathbb{Z}} \frac{1}{\pi(1+n^2)} = \pi \sum_{k\in\mathbb{Z}} e^{-2\pi|k|}$$

$$= \pi \left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right)$$

Also, since $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 1$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx \underset{f \text{ is real}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot f(x)\,dx \underset{0\le f(x)\le\pi}{\le} \frac{1}{2\pi}\cdot\pi\int_{-\pi}^{\pi} f(x)\,dx = \pi$$

In (*) we obtain:

$$\pi < \pi\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right) \le \pi$$

, that is a contradiction. Then, f doesn't exist.

Problem 3

a) Let $f(x) = (x^2 + 2x + 2)e^{-\pi x^2}$. Compute the Fourier transform of f.

b) Let $g : \mathbb{R} \to \mathbb{R}$ be the function such that

$$e^{\pi t^2} g(t) = \int_{-\infty}^{\infty} (x^2 + 2x + 2)e^{-2\pi x(x-t)}\, dx,$$

for all $t \in \mathbb{R}$. Compute the Fourier transform of g.

SOLUTION:

a) We remark that, if $f \in \mathcal{M}(\mathbb{R})$ and $xf \in \mathcal{M}(\mathbb{R})$, we have $\widehat{-2\pi i x f}(\xi) = (\hat{f})'(\xi)$. Then

$$\hat{f} = \widehat{x^2 e^{-\pi x^2}} + 2\widehat{xe^{-\pi x^2}} + 2\widehat{e^{-\pi x^2}} = \frac{1}{-2\pi i}\left(\widehat{-2\pi i x \cdot x e^{-\pi x^2}}\right) + \frac{1}{-\pi i}\left(\widehat{-2\pi i x\, e^{-\pi x^2}}\right) + 2e^{-\pi x^2}$$

$$= \frac{1}{-2\pi i}\left(\widehat{x e^{-\pi x^2}}\right)' + \frac{1}{(-\pi i)}\left(\widehat{e^{-\pi x^2}}\right)' + 2e^{-\pi x^2} = \frac{1}{(-2\pi i)^2}\left(\widehat{-2\pi i x\, e^{-\pi x^2}}\right)' + \frac{1}{(-\pi i)}\left(e^{-\pi x^2}\right)' + 2e^{-\pi x^2}$$

$$= \frac{-1}{4\pi^2}\left(\widehat{e^{-\pi x^2}}\right)'' + \frac{1}{(-\pi i)}\left(e^{-\pi x^2}\right)' + 2e^{-\pi x^2} = \frac{-1}{4\pi^2}\left(e^{-\pi x^2}\right)'' - \frac{1}{\pi i}\left(e^{-\pi x^2}\right)' + 2e^{-\pi x^2}$$

$$\Rightarrow \hat{f}(x) = \left(-x^2 - 2xi + \left(2 + \frac{1}{2\pi}\right)\right)e^{-\pi x^2}$$

b)

$$g(t) = \int_{-\infty}^{\infty} (x^2+2x+2)\, e^{-2\pi x^2 + 2\pi x t - \pi t^2}\, dx = \int_{-\infty}^{\infty} (x^2+2x+2)\, e^{-\pi x^2} \cdot e^{-\pi(x^2 - 2xt + t^2)}\, dx$$

$$g(t) = \int_{-\infty}^{\infty} f(x)\, e^{-\pi(t-x)^2}\, dx \quad \Rightarrow \quad g = f * e^{-\pi x^2} \Rightarrow \hat{g}(t) = \hat{f}(t)\, e^{-\pi t^2}$$

Using a) we conclude.

Problem 4 Let $m, n \geq 0$ be integers numbers. Suppose that for any function $G \in C^{\infty}(\mathbb{R})$ with compact support the following inequality holds[1] :

$$\sup_{\xi \in \mathbb{R}} \left| \xi^m \widehat{G}(\xi) \right| \leq \int_{-\infty}^{\infty} |G^{(n)}(x)|\, dx.$$

Prove that $m = n$.

SOLUTION: Let $f \in C^{\infty}(\mathbb{R})$ with compact support, no identically zero. Let $\delta > 0$, and define $G(x) = f(\delta x)$. Note that $\widehat{G}(\xi) = \frac{1}{\delta} \widehat{f}\left(\frac{\xi}{\delta}\right)$

and $G^{(n)}(x) = \delta^n f^{(n)}(\delta x) \quad \forall n \geq 0.$

Since the inequality holds for G, we have:

$$\sup_{\xi \in \mathbb{R}} \left| \xi^m \cdot \frac{1}{\delta} \widehat{f}\left(\frac{\xi}{\delta}\right) \right| \leq \int_{-\infty}^{\infty} |\delta^n f^{(n)}(\delta x)|\, dx$$

Now: $\sup_{\xi \in \mathbb{R}} \left| \xi^m \cdot \frac{1}{\delta} \widehat{f}\left(\frac{\xi}{\delta}\right) \right| = \sup_{\xi \in \mathbb{R}} \left| \left(\frac{\xi}{\delta}\right)^m \delta^{m-1} \widehat{f}\left(\frac{\xi}{\delta}\right) \right| = \delta^{m-1} \sup_{\xi \in \mathbb{R}} \left| \left(\frac{\xi}{\delta}\right)^m \widehat{f}\left(\frac{\xi}{\delta}\right) \right| = \delta^{m-1} \sup_{\xi \in \mathbb{R}} |\xi^m \widehat{f}(\xi)|.$

Moreover: $\int_{-\infty}^{\infty} |\delta^n f^{(n)}(\underbrace{\delta x}_{\delta x = u})|\, dx = \int_{-\infty}^{\infty} |\delta^n f^{(n)}(u)| \frac{du}{\delta} = \delta^{n-1} \int_{-\infty}^{\infty} |f^{(n)}(u)|\, du.$

Therefore: $\delta^{m-1} \sup_{\xi \in \mathbb{R}} |\xi^m \widehat{f}(\xi)| \leq \delta^{n-1} \int_{-\infty}^{\infty} |f^{(n)}(u)|\, du \Rightarrow \sup_{\xi \in \mathbb{R}} |\xi^m \widehat{f}(\xi)| \leq \delta^{n-m} \int_{-\infty}^{\infty} |f^{(n)}(u)|\, du$

If: $n > m \Rightarrow$ letting $\delta \to 0$, we get that $\sup_{\xi \in \mathbb{R}} |\xi^m \widehat{f}(\xi)| = 0 \Rightarrow \widehat{f}(\xi) = 0 \ \forall \xi \Rightarrow f \equiv 0$ contradiction

If: $n < m \Rightarrow$ letting $\delta \to \infty$, and using the above argument we obtain a contradiction.

We conclude that $n = m$.

[1]The function $G^{(n)}$ denotes the n-th derivative of G.

Problem 5 Let $\mathcal{F}$ be the family of entire functions $f : \mathbb{C} \to \mathbb{C}$ of exponential type 2π such that $f \in \mathcal{M}(\mathbb{R})$, $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $f(0) = 1$.

a) Prove that for any $f \in \mathcal{F}$ we have

$$\int_{-\infty}^{\infty} f(x)\, dx \geq 1.$$

b) Give an example of a function $f \in \mathcal{F}$ such that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{1}$$

c) Prove that there exists a unique function $f \in \mathcal{F}$ such that (1) holds.

SOLUTION:

a) Since $f \in \mathcal{M}(R)$, using Paley Wiener theorem, $\operatorname{supp}(\hat{f}) \subset [-1,1] \Rightarrow \hat{f} \in \mathcal{M}(R)$.

Using Poisson Summation formula:

$$\int_{-\infty}^{\infty} f(x)dx = \hat{f}(0) = \sum_{n\in \mathbb{Z}} \hat{f}(n) = \sum_{m\in\mathbb{Z}} f(m) = f(0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f(m) \geq f(0) = 1 \ldots (*)$$

b) $f(x) = \left(\dfrac{\sin \pi x}{\pi x}\right)^2$

c) Note that if $\int_{-\infty}^{\infty} f(x)dx = 1$, then in $(*)$ we obtain that $f(m) = 0 \ \forall\, m \in \mathbb{Z}\setminus\{0\}$.

CLAIM: $f'(n) = 0 \quad \forall n \in \mathbb{Z}\setminus\{0\}$.

Let $n \in \mathbb{Z}\setminus\{0\}$ and $x > n$. We have $\dfrac{f(x) - f(n)}{x-n} = \dfrac{f(x)}{x-n} \geq 0 \underset{x\to n^+}{\Rightarrow} f'(n) \geq 0$

When $x < n$ we have $\dfrac{f(x) - f(n)}{x-n} = \dfrac{f(x)}{x-n} \leq 0 \Rightarrow_{x\to n^-} f'(n) \leq 0$.

We conclude that $f'(n) = 0 \quad \forall\, n \in \mathbb{Z}\setminus\{0\}$

Now, using Vaaler's interpolation formula, we obtain that

$$f(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \frac{f(0)}{x^2} + \frac{f'(0)}{x} \right\} = \left(\frac{\sin\pi x}{\pi}\right)^2 \left\{ \frac{1}{x^2} + \frac{f'(0)}{x} \right\} = \left(\frac{\sin \pi x}{\pi x}\right)^2 + \frac{f'(0)}{\pi^2}\frac{(\sin \pi x)^2}{x}$$

Since $f \in \mathcal{M}(R)$ and $\left(\dfrac{\sin\pi x}{\pi x}\right)^2 \in \mathcal{M}(R) \Rightarrow \dfrac{f'(0)}{\pi^2}\dfrac{(\sin \pi x)^2}{x} \in \mathcal{M}(R)$.

Note that $\exists\, A > 0$ and $\delta > 0$: $\left|\dfrac{f'(0)}{\pi^2}\dfrac{(\sin\pi x)^2}{x}\right| \leq \dfrac{A}{|x|^{1+\delta}}$, for $x \to \infty$. Taking $x_k = \dfrac{(2k+1)}{2}$

$$\Rightarrow \frac{|f'(0)|}{\pi^2} \leq \frac{A}{\left|\frac{2k+1}{2}\right|^{\delta}}.$$ Letting $k \to \infty \Rightarrow f'(0) = 0$. Therefore

$$f(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$$