Paley–Wiener Thm

Thm

Let $f \in \mathcal{M}(\mathbb{R})$. TFAE

a) f has an analytic extension on $\mathbb{C}$ s.t.

$$|f(z)| \leq A e^{2\pi M |z|}, \quad \forall\, z \in \mathbb{C} \text{ for some } A, M > 0.$$

b) $\hat{f}(t) = 0, \quad \forall\, |t| \geq M$

Lemma

Let f holomorphic in $\{\operatorname{Im} s < a\}$, with $a > 0$.

Also:

$$|f(x+iy)| \leq \frac{A e^{2\pi M |y|}}{1 + |x|^{1+\delta}} \quad \text{for } -b \leq y \leq 0$$

(some $b > 0$)

Then:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x - ib) e^{-2\pi i t (x - ib)}\, dx, \text{ for } t \geq M$$

Lemma: Let f holomorphic in $\{\operatorname{Im} s < a\}$ with $a > 0$.

Suppose:

$$|f(x+iy)| \le \frac{A e^{2\pi M|y|}}{1+|x|^{1+\delta}}, \quad y<0$$

Then $\hat{f}(t)=0$, for $t \geq M$

Proof

Assume $t > M$. Let $b>0$.

Then, when $-b \le y \le 0$:

$$|f(x+iy)| \le \frac{\overbrace{Ae^{2\pi M b}}^{=\tilde{A}}}{1+|x|^{1+\delta}} \le \frac{\tilde{A} e^{2M|y|}}{1+|x|^{1+\delta}}$$

$$|\hat{f}(t)| \le \int_{-\infty}^{\infty} \frac{Ae^{2\pi M|b|}}{1+|x|^{1+\delta}} e^{-2\pi t b}\, dx$$

$$= e^{2\pi b(M-t)} \underbrace{\int_{-\infty}^{\infty} \frac{A}{1+|x|^{1+\delta}}\, dx}_{K}$$

$$|\hat{f}(t)| \leqslant K e^{-2\pi b(t-M)}$$

Letting $b \to \infty \Rightarrow \hat{f}(t) = 0$

Lemma:

Let f be an entire function s.t. $f \in M(\mathbb{R})$

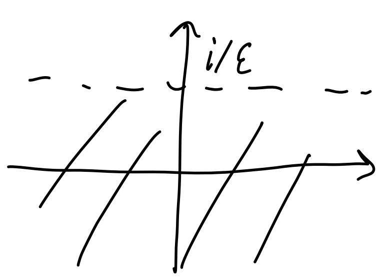
Suppose that

$$|f(x+iy)| \leqslant A e^{2\pi M|y|}, \text{ for all } x \in \mathbb{R}, y \in \mathbb{R}$$

Then $\hat{f}(t) = 0$ for $t \geqslant M$

Proof:

Let $0 < \varepsilon < 1$. Let $f_\varepsilon(z) = \dfrac{f(z)}{(1+i\varepsilon z)^2}$



↑ pole $\frac{i}{\varepsilon}$

f_ε is holomorphic in $\{\operatorname{Im} s < \frac{1}{\varepsilon}\}$

$$|f_\varepsilon(x+iy)| \leq \frac{A e^{2\pi M|y|}}{|1+i\varepsilon(x+iy)|^2} = \frac{A e^{2\pi M|y|}}{(1-\varepsilon y)^2+(\varepsilon x)^2}$$

$$= \frac{\frac{A}{\varepsilon^2} e^{2\pi M|y|}}{(\frac{1}{\varepsilon}-y)^2+x^2} \leq *$$

For $y \leq 0 \Rightarrow \frac{1}{\varepsilon} - y \geq \frac{1}{\varepsilon} > 1$

$$* \leq \frac{\frac{A}{\varepsilon^2} e^{2\pi M|y|}}{1+x^2}$$

$\Rightarrow \widehat{f_\varepsilon}(t) = 0$, for $t \geq M$

Let us prove that $\hat{f}_\varepsilon(t) \xrightarrow[\varepsilon \to 0^+]{} \hat{f}(t)$

$$\left|\hat{f}_\varepsilon(t) - \hat{f}(t)\right| \leq \int_{-\infty}^{\infty} |f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| dx$$

$$= \int_{|x| \geq \frac{1}{\varepsilon^{1/8}}} |f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| dx$$

$$+ \int_{|x| \leq \frac{1}{\varepsilon^{1/8}}} |f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| dx = I_1 + I_2$$

Bounds:

$$\left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| \leq \frac{1}{|1+i\varepsilon x|^2} + 1 \leq 2$$

$$\left|\frac{1}{(1+i\varepsilon x)^2}-1\right| = \left|\frac{1-(1+i\varepsilon x)^2}{(1+i\varepsilon x)^2}\right|$$

$$= \frac{\left|-2i\varepsilon x+\varepsilon^2 x^2\right|}{1+\varepsilon^2 x^2} = \frac{\varepsilon\sqrt{\left|-2ix+\varepsilon x^2\right|^2}}{1+\varepsilon^2 x^2}$$

$$= \frac{\varepsilon\sqrt{\varepsilon^2 x^4+4x^2}}{1+\varepsilon^2 x^2} = \varepsilon|x|\overbrace{\frac{\sqrt{\varepsilon x^2+4}}{1+\varepsilon^2 x^2}}^{<2} < 2\varepsilon|x|$$

$$I_1 \leq 2\int_{|x|\geq \frac{1}{\varepsilon^{1/8}}} |f(x)|\,dx \Rightarrow \text{When } \varepsilon\to 0,\ I_1 \to 0$$

(since $f\in M(\mathbb{R})$)

$$I_2 \leq \int_{|x|\leq\frac{1}{\varepsilon^{1/8}}} B\,2\varepsilon|x|\,dx = 2\varepsilon B\left[2\int_0^{1/\varepsilon^{1/8}} x\,dx\right]$$

$$= 2\varepsilon B \cancel{\chi}\left[\frac{\left(\frac{1}{\varepsilon^{1/8}}\right)^2}{\cancel{\chi}}\right] = 2B\varepsilon^{3/4}$$

when $\varepsilon \to 0 \Rightarrow I_2 \to 0$ (we are done)

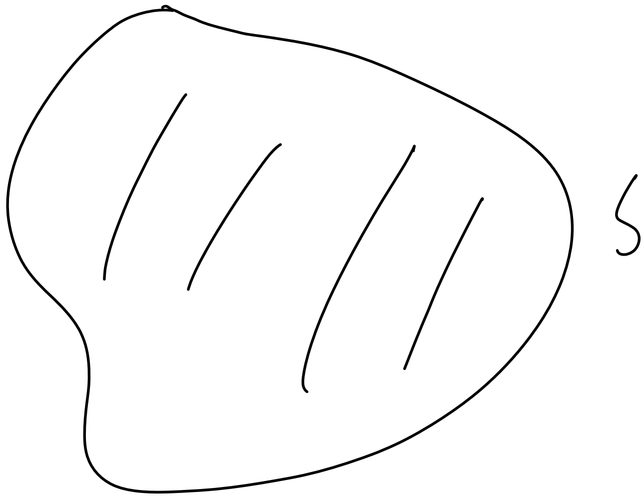
<u>Phargmén Lindelöf thm:</u>

Let f be holomorphic in S where

$S = \{ z \in \mathbb{C} \mid -\frac{\pi}{4} < \arg z < \frac{\pi}{4} \}$. Suppose that:

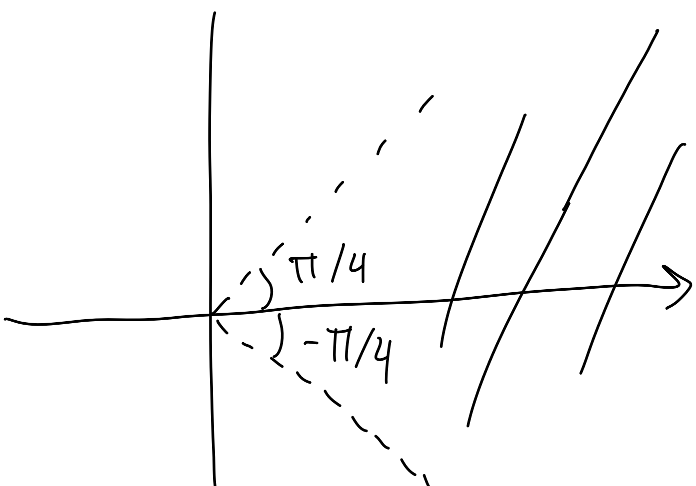
$$\left.\begin{array}{l} *)\ f \text{ is Continuous in } \overline{S} \\ *)\ |f(z)| \leq C e^{B|z|} \text{ for } z \in S \\ *)\ |f(z)| \leq N \ \forall\ z \in \partial S \end{array}\right\} \Rightarrow \begin{array}{l} |f(z)| \leq N \\ \text{for } z \in S \end{array}$$

Maximum-Modulus principle:



$$|f(z)| \le N \quad \forall\ z \in \partial S$$

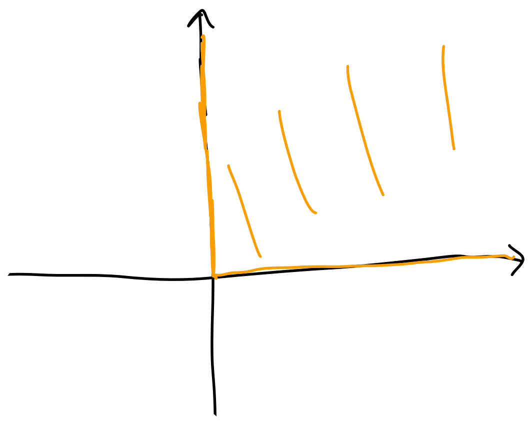
$$\Rightarrow |f(z)| \le N \quad \forall\ z \in S$$



(can use $G(z) = f(e^{i\pi/4} z)$ to rotate the sector)

Proof of thm P.W:

Define $g(z) = f(z) e^{2\pi i z M}$



$$|g(z)| \leq |f(z)| e^{|2\pi i z M|}$$

$$\leq A e^{2\pi M |z|} e^{2\pi M |z|}$$

$$= A e^{4\pi i M |z|}$$

Let $x > 0$:

$$g(x) = f(x) e^{2\pi i x M}$$

$$|g(x)| = |f(x)| \leq \frac{B}{N} \leq \frac{B+A}{N} \quad \forall\, x \geq 0$$

Let $y > 0$:

$$|g(y)| = |f(iy)|\,\left|e^{2\pi i (iy) M}\right|$$

$$\leq \frac{A}{N} e^{2\pi M|y|} e^{-2\pi y M} = \frac{A}{N} e^{2\pi M y} e^{-2\pi y M} = \frac{A}{N} \leq \frac{B+A}{N}$$

Applying Phragmén Lindelöf thm, we have

$$|g(z)| \leq N \quad \forall \; z \in$$

$$\left|f(z) e^{2\pi i (x+iy) M}\right| \leq N$$

$$|f(x+iy)| \leq N e^{2\pi |y| M}, \quad \text{for} \quad \begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}$$

$\Rightarrow$ The same idea works in other quadrants.

$\Rightarrow$ $|f(x+iy)| \leq \tilde{N} e^{2\pi M|y|}$ for all $x, y \in \mathbb{R}$

Using the previous lemma

$\Rightarrow$ $\hat{f}(t) = 0$ for $t \geq M$

Define $g(x) = f(-x)$ $\Rightarrow$ $\hat{g}(t) = 0$ $t \geq M$

$\hat{f}(-t) = 0$ $t \geq M$

$\hat{f}(t) = 0$ $t \leq -M$

$\Rightarrow$ $\hat{f}(t) = 0, \; \forall \; |t| \geq M$

$$f(z) = \left(\frac{\sin \pi z}{\pi z} \right)^2$$

$$|f(z)| \leq A e^{2\pi M |z|} \quad \forall \ z \geq N$$

$$|z| \leq N: \quad |f(z)| \leq L \leq L e^{2\pi M |z|}$$

$$\Rightarrow \quad |f(z)| \leq (A+L) e^{2\pi M |z|} \quad \forall \ z \in \mathbb{C}$$

$|z| \geq 1$:

$$|f(z)| \leq \frac{|\sin \pi z|^2}{\pi^2} = \left| \frac{e^{\pi i z} - e^{-\pi i z}}{2i} \right| \frac{1}{\pi^2}$$

$$\leq \frac{\left(|e^{\pi i z}| + |e^{-\pi i z}| \right)^2}{4\pi^2} \leq \frac{2\left(|e^{\pi i z}|^2 + |e^{\pi i z}|^2 \right)}{4\pi^2}$$

$\leq Ce^{2\pi|z|}$

$\Rightarrow \quad \hat{f}(t)=0 \quad \forall \ |t|\geq 1$

$|f(z)| \leq Ae^{2\pi M|z|} \Rightarrow$ f has exponential type $2\pi M$

Theorem:

Let f be a function of exponential type π, $f\in M(\mathbb{R})$. Then:

$$\sum_{n\in\mathbb{Z}} |f(n)|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

Corollary:

Under the same assumptions:

$$f(n) = 0 \quad \forall\, n \in \mathbb{Z} \implies f \equiv 0$$