Fourier analysis exercise 54, 55, 56a

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Exercise 54)

We first observe that

$$\frac{a^2-x^2}{(a^2+x^2)^2} = \frac{a^2+x^2-x^2-x^2}{(a^2+x^2)^2} = \frac{1}{a^2+x^2} - 2\frac{x^2}{(a^2+x^2)^2} = \frac{\pi}{a}P_a(x) - 2\mathcal{T}_a(x)$$

In the last line we define $\mathcal{T}_a$ implicitly. We know from problem 52 that

$$\mathcal{F}\left\{\frac{\pi}{a}P_a\right\}(\xi) = \frac{\pi}{a}e^{-2\pi|\xi|a}$$

so what is left is to find $\mathcal{F}\{\mathcal{T}_a\}$. We start by observing that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{a^2+x^2} = \frac{-2x}{(a^2+x^2)^2}$$

And hence we have

$$\frac{x^2}{(a^2+x^2)^2} = \frac{1}{4\pi i}\left(-2\pi i x \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{a^2+x^2}\right)$$

Why is this nice? Recall that we under certain assumptions have the following formulas for $f \in \mathcal{M}(\mathbb{R})$:

$$\mathcal{F}\{-2\pi i x f(x)\} = \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}\{f\}(\xi) \qquad \mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right\}(\xi) = 2\pi i\xi\mathcal{F}\{f\}(\xi)$$

For the first formula to work we need the additional assumption that $xf(x) \in \mathcal{M}(\mathbb{R})$ and for the second formula we need $f \in \mathcal{M}(\mathbb{R}) \cap C^1(\mathbb{R})$ and $f' \in \mathcal{M}(\mathbb{R})$. It is not hard to see that all this assumptions are furfilled in the application of the formulas under:

$$\begin{aligned}
\mathcal{F}\{\mathcal{T}_a(x)\}(\xi) &= \mathcal{F}\left\{\frac{1}{4\pi i}\left(-2\pi i x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{a^2+x^2}\right)\right\} = \frac{1}{4\pi i}\mathcal{F}\left\{\left(-2\pi i x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{a^2+x^2}\right)\right\} \\
&= \frac{1}{4\pi i}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{a^2+x^2}\right\} = \frac{\pi}{4\pi i a}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\pi}\frac{a}{a^2+x^2}\right\} \\
&= \frac{1}{4ia}\frac{\mathrm{d}}{\mathrm{d}\xi}2\pi i\xi\mathcal{F}\{P_a(x)\}(\xi) \\
&= \frac{\pi}{2a}\frac{\mathrm{d}}{\mathrm{d}\xi}\xi e^{-2\pi|\xi|a} = \frac{\pi}{2a}\frac{e^{-2\pi a|\xi|}(|\xi| - 2\pi a\xi^2)}{|\xi|}
\end{aligned}$$

Hence we finally conclude that

$$\mathcal{F}\left\{\frac{a^2-x^2}{(a^2+x^2)^2}\right\} = \frac{\pi}{a}e^{-2\pi|\xi|a} - 2\frac{\pi}{2a}\frac{e^{-2\pi a|\xi|}(|\xi|-2\pi a|\xi|^2)}{|\xi|} = 2\pi^2|\xi|e^{-2\pi a|\xi|}$$

Exercise 55)

We observe

$$\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\,\mathrm{d}y = \int_{-\infty}^{\infty} f(y)e^{2xy-y^2}\,\mathrm{d}y = \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2+x^2}\,\mathrm{d}y = e^{x^2}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2}\,\mathrm{d}y$$

Since $e^{x^2}\neq 0$ we conclude that $\forall x\in\mathbb{R}$ we have

$$0 = \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2}\,\mathrm{d}y = (f * e^{-y^2})(x)$$

Applying the Fourier transform to both sides and the convolution rule for Fourier transform we get that $0 = \widehat{f}(\xi)\widehat{e^{-x^2}}(\xi)$, and the Fourier transform of the Gaussian is never zero so $\widehat{f}(\xi)\equiv 0$ so by problem 47c) we can conclude that $f\equiv 0$.

Exercise 56)

a)

First observe that $h=0$ is trivial so we may assume $h\neq 0$. We apply Fourier inversion (which we can since $f,\widehat{f}\in\mathcal{M}(\mathbb{R})$) to get

$$\begin{aligned}
|f(x+h)-f(x)| &= \left|\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{2\pi ix\xi}\left(e^{2\pi ih\xi}-1\right)\,\mathrm{d}\xi\right| \\
&\leq \left|\int_{|\xi|\leq\frac{1}{|h|}}\widehat{f}(\xi)e^{2\pi ix\xi}\left(e^{2\pi ih\xi}-1\right)\,\mathrm{d}\xi\right| + \left|\int_{|\xi|>\frac{1}{|h|}}\widehat{f}(\xi)e^{2\pi ix\xi}\left(e^{2\pi ih\xi}-1\right)\,\mathrm{d}\xi\right| \\
&= \mathcal{I}_1+\mathcal{I}_2
\end{aligned}$$

Now the bounding begins. Since $\widehat{f}\in\mathcal{M}(\mathbb{R})$, $\widehat{f}(\xi) = O(|\xi|^{-1-\alpha})$. Since $\xi^{-\zeta}\geq\xi^{-\eta}$ when $0\leq\zeta\leq\eta$, there is no harm in assuming $0<\alpha<1$. Bounding $\mathcal{I}_2$ we get for some constant C that:

$$\mathcal{I}_2 \leq 2\int_{|\xi|>\frac{1}{|h|}}|\widehat{f}(\xi)|\,\mathrm{d}\xi \overset{\widehat{f}\in\mathcal{M}(\mathbb{R})}{\leq} 2C\int_{|\xi|>\frac{1}{|h|}}\frac{1}{|\xi|^{1+\alpha}}\,\mathrm{d}\xi = 4C\left[-\frac{1}{\alpha}\frac{1}{\xi^{\alpha}}\right]_{\frac{1}{|h|}}^{\infty} \leq \frac{4C}{\alpha}|h|^{\alpha}$$

To bound $\mathcal{I}_1$ we split up into two cases: $|h|<1$ and $|h|\geq 1$. By Problem 45 we can find a constant K such that

$$|\widehat{f}(\xi)|\leq\frac{K}{1+|\xi|^{1+\alpha}}\qquad\forall\xi\in\mathbb{R}$$

Let us first consider the case $|h| \geq 1$: using the same bounds as for $\mathcal{I}_2$ and a trivial estimate we get

$$\mathcal{I}_2 \leq 2K \int_{|\xi| \leq \frac{1}{|h|}} \frac{1}{1+|\xi|^{1+\alpha}} \, \mathrm{d}\xi = 4K \frac{1}{|h|}$$

Since $|h| \geq 1$ we have $\frac{1}{|h|} \leq |h|^\alpha$, which finish the bound in this case, because then

$$\mathcal{I}_1 + \mathcal{I}_2 \leq \left(4K + \frac{4C}{\alpha}\right) |h|^\alpha$$

What we need to finish the proof is to do the case $|h| < 1$. To do this we split up the integral once more: into the integral over $|\xi| \leq 1$ and $1 < |\xi| < \frac{1}{|h|}$. Let us call those integrals $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively. For $\mathcal{J}_1$, recall the bound proven in class: $|e^{2\pi i \xi h} - 1| \leq |2\pi \xi h|$. Since $\widehat{f}$ is continuous (since it is in $\mathcal{M}(\mathbb{R})$) and $|\xi| \leq 1$ is a compact set, we find a constant $C_f > 0$ such that $|\widehat{f}(\xi)| \leq C_f$ for all $|\xi| \leq 1$. Then

$$|\mathcal{J}_1| \leq \int_{|\xi| \leq 1} |2\pi \xi h| C_f \, \mathrm{d}\xi \leq 4\pi C_f |h|$$

To bound $\mathcal{J}_2$ we simply integrate and use that $|\widehat{f}(\xi)| \leq \frac{M}{|\xi|^{1+\alpha}}$ for some constant $M > 0$ when $1 < |\xi| < \frac{1}{|h|}$. Now

$$\begin{aligned}
|\mathcal{J}_2| &\leq \int_{1 < |\xi| \leq \frac{1}{|h|}} |2\pi \xi h| |\widehat{f}(\xi)| \, \mathrm{d}\xi \\
&\leq 2\pi |h| M \int_{1 < |\xi| \leq \frac{1}{|h|}} \frac{1}{|\xi|^\alpha} \, \mathrm{d}\xi \\
&= \frac{4\pi |h| M}{1-\alpha} \left[\frac{1}{\xi^{\alpha-1}}\right]_1^{|h|^{-1}} = \frac{4\pi |h| M}{1-\alpha} (|h|^{\alpha-1} - 1) \\
&\leq \frac{4\pi |h| M}{1-\alpha} |h|^{\alpha-1} = \frac{4\pi M}{1-\alpha} |h|^\alpha
\end{aligned}$$

Hence we have in the case that $|h| < 1$ found a constant

$$D = 4\pi C_f + \frac{4\pi M}{1-\alpha}$$

such that

$$\mathcal{I}_1 \leq D(|h|^\alpha + |h|) \leq D(|h|^\alpha + |h|^\alpha) \leq 2D|h|^\alpha$$

where we in the last inequality have used that $|h| < 1$ and $0 < \alpha < 1$. Thus we can also conclude in the case $|h| < 1$ that

$$\mathcal{I}_1 + \mathcal{I}_2 \leq \left(2D + \frac{4C}{\alpha}\right) |h|^\alpha$$

so we have proved that f satisifies the Hölder condition with exponent α.