

Lecture notes

Continuous log's and winding numbers. Part 2.

1. WINDING NUMBERS OF JORDAN CURVES.

Theorem 1 (without proof). *Let γ be a Jordan curve in $\mathbb{C}$. Then:*

- (i) $\text{ind}(\gamma, z_0) = 0$ for any $z_0 \in \text{ext}(\gamma)$;
- (ii) $\text{ind}(\gamma, z_0)$ either equals 1 for all $z_0 \in \text{int}(\gamma)$, or it is equal to -1 for all $z_0 \in \text{int}(\gamma)$.

Definition 1. A Jordan curve γ is said to be *oriented counterclockwise*, if $\text{ind}(\gamma, z_0) = 1$ for all $z_0 \in \text{int}(\gamma)$. Otherwise, i.e. if $\text{ind}(\gamma, z_0) = -1$ for all $z_0 \in \text{int}(\gamma)$, we say that γ is oriented *clockwise*.

2. HOMOTOPY.

Definition 2. Closed curves $\gamma_1 : [0, 1] \rightarrow \mathbb{C}$ and $\gamma_2 : [0, 1] \rightarrow \mathbb{C}$ are said to be *homotopic in a domain D* , if there exists a continuous mapping $\gamma : [0, 1] \times [0, 1] \rightarrow D$ such that:

- (i) $\gamma(t, 0) = \gamma_1(t)$ for all $t \in [0, 1]$;
- (ii) $\gamma(t, 1) = \gamma_2(t)$ for all $t \in [0, 1]$;
- (iii) $\gamma(0, \theta) = \gamma(1, \theta)$ for all $\theta \in [0, 1]$.

The mapping γ is called a *homotopical family* or *homotopy* (for γ_1 and γ_2).

Remark 1. The conditions (i) and (ii) in the above definition implies that the curves γ_1 and γ_2 lie in the domain D .

The condition (iii) means that all the intermediate curves $t \mapsto \gamma(t, \theta)$, $0 < \theta < 1$, are also required to be *closed*.

3. HOMOTOPY AND WINDING NUMBERS.

Here we will prove that homotopic curves has the same winding numbers.

Let us start with some elementary question.

Let $z, w \in \mathbb{C} \setminus (-\infty, 0]$. Is it true that

$$\text{Arg } z + \text{Arg } w = \text{Arg}(zw) \quad \text{and} \quad \text{Arg } z - \text{Arg } w = \text{Arg}(z/w)?$$

Not always! Of course, $\text{Arg } z + \text{Arg } w$ is one of the values of $\arg(zw)$, but not necessarily the principal value. For example, take $w = z = e^{2\pi i/3}$. Then $\text{Arg}(w) = \text{Arg}(z) = 2\pi/3$, but $\text{Arg}(zw) = \text{Arg}(e^{4\pi i/3}) = -2\pi/3$. A similar example can be constructed for $\text{Arg } z - \text{Arg } w$.

However, if $\text{Arg } z$ and $\text{Arg } w$ are less than $\pi/2$ by the absolute value, then $\text{Arg } z \pm \text{Arg } w$ belongs to $(-\pi, \pi)$ and hence $\text{Arg } z - \text{Arg } w$ and $\text{Arg } z + \text{Arg } w$ are the *principal* values of

$\arg(z/w)$ and $\arg(zw)$, respectively. Recall that $\text{Log } z = \ln |z| + i \text{Arg } z$. Therefore, we conclude that the formulas $\text{Log } z + \text{Log } w = \text{Log}(zw)$ and $\text{Log } z - \text{Log } w = \text{Log}(zw)$ are valid for any z and w in the right half-plane $\{\zeta : \text{Re } \zeta > 0\}$. For arbitrary $z, w \in \mathbb{C} \setminus \{0\}$ these formulas can fail!

Now we can prove the following statement.

Theorem 2. *Let $z_0 \in \mathbb{C}$ and γ_1, γ_2 two closed curves homotopic in $\mathbb{C} \setminus \{z_0\}$. Then*

$$\text{ind}(\gamma_1, z_0) = \text{ind}(\gamma_2, z_0).$$

Proof. Let $\gamma : [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{z_0\}$ be a homotopical family for γ_1 and γ_2 .

Denote $R := \text{dist}(\gamma([0, 1] \times [0, 1]), z_0) > 0$.

Divide the rectangle $[0, 1] \times [0, 1]$ into N^2 equal parts. For $k, j = 0, \dots, N$ we denote

$$t_k := k/N,$$

$$\theta_j := j/N,$$

$$z_{k,j} := \gamma(t_k, \theta_j),$$

$$D_{k,j} := \{z : |z - z_{k,j}| < R\}.$$

According to the Heine–Cantor theorem, we may assume that N is so large that for any $k, j = 0, \dots, N-1$, we have $\gamma([t_k, t_{k+1}] \times [\theta_j, \theta_{j+1}]) \subset D_{k,j}$.

For $j = 0, \dots, N$, denote by γ^j the intermediate curve in the homotopical family corresponding to $\theta = \theta_j$, i.e. $\gamma^j(t) := \gamma(t, \theta_j)$, $t \in [0, 1]$. Clearly, it is sufficient to prove that $\text{ind}(\gamma^j, z_0) = \text{ind}(\gamma^{j+1}, z_0)$ for all $j = 0, \dots, N-1$, because $\gamma_1 = \gamma^0$ and $\gamma_2 = \gamma^N$.

By definition,

$$\text{ind}(\gamma^j, z_0) = \frac{1}{2\pi i} \Delta(\log(z - z_0), \gamma^j) = \frac{1}{2\pi i} (\psi_j(1) - \psi_j(0)),$$

where $\psi_j : [0, 1] \rightarrow \mathbb{C}$ is a continuous logarithm along the curve $\tilde{\gamma}^j$ defined by $\tilde{\gamma}^j(t) = \gamma^j(t) - z_0$.

Fix now any $k = 0, \dots, N-1$. Consider the part $\tilde{\gamma}^{j,k}$ of the curve $\tilde{\gamma}^j$ corresponding to the values of t in $[t_k, t_{k+1}]$. That is, $\tilde{\gamma}^{j,k} = \tilde{\gamma}^j|_{[t_k, t_{k+1}]}$. By construction, $\gamma(t, \theta_j) \in D_{k,j}$ for $t \in [t_k, t_{k+1}]$. Therefore, the curve $\tilde{\gamma}^{j,k}$ lies in the disk $\tilde{D}_{k,j} := \{z - z_0 : z \in D_{k,j}\} = \{z : |z - (z_{k,j} - z_0)| < R\}$. Clearly, the restriction $\psi_j|_{[t_k, t_{k+1}]}$ of ψ_j to $[t_k, t_{k+1}]$ is a continuous logarithm along $\tilde{\gamma}^{j,k}$. Note that $0 \notin \tilde{D}_{k,j}$ because $z_0 \notin D_{k,j}$ by construction. Then from the last statement of the Theorem on Continuous Logarithms, for $t \in [t_k, t_{k+1}]$ we have

$$\psi_j(t) = \psi_j(t_k) + \text{Log} \frac{\tilde{\gamma}^{j,k}(t)}{\tilde{\gamma}^{j,k}(t_k)} = \psi_j(t_k) + \text{Log} \frac{\gamma^j(t) - z_0}{\gamma^j(t_k) - z_0}.$$

In particular, for $t = t_{k+1}$ we have

$$\psi_j(t_{k+1}) - \psi_j(t_k) = \text{Log} \frac{\gamma^j(t_{k+1}) - z_0}{\gamma^j(t_k) - z_0} = \text{Log} \frac{z_{k+1,j} - z_0}{z_{k,j} - z_0}.$$

Summing w.r.t. $k = 0, \dots, N-1$, we get

$$\psi_j(1) - \psi_j(0) = \sum_{k=0}^{N-1} \text{Log} \frac{z_{k+1,j} - z_0}{z_{k,j} - z_0}.$$

Hence

$$(1) \quad \psi_{j+1}(1) - \psi_{j+1}(0) - (\psi_j(1) - \psi_j(0)) = \sum_{k=0}^{N-1} \left(\text{Log} \frac{z_{k+1,j+1} - z_0}{z_{k,j+1} - z_0} - \text{Log} \frac{z_{k+1,j} - z_0}{z_{k,j} - z_0} \right)$$

We claim that

$$(2) \quad \text{Log} \frac{z_{k,j+1} - z_0}{z_{k,j} - z_0} + \text{Log} \frac{z_{k+1,j+1} - z_0}{z_{k,j+1} - z_0} + \text{Log} \frac{z_{k+1,j} - z_0}{z_{k+1,j+1} - z_0} + \text{Log} \frac{z_{k,j} - z_0}{z_{k+1,j} - z_0} = 0.$$

Indeed, denote

$$\zeta_0 := z_{k,j} - z_0, \quad \zeta_1 := z_{k,j+1} - z_0, \quad \zeta_2 := z_{k+1,j+1} - z_0, \quad \zeta_3 := z_{k+1,j} - z_0.$$

Then (2) can be rewritten as $\text{Log}(\zeta_1/\zeta_0) + \text{Log}(\zeta_2/\zeta_1) + \text{Log}(\zeta_3/\zeta_2) + \text{Log}(\zeta_0/\zeta_3) = 0$. Note that by construction, $\zeta_m \in \tilde{D}_{k,j} \not\equiv 0$ for all $m = 0, \dots, 3$. The point $z_{k,j} - z_0$ is the center of the disk $\tilde{D}_{k,j}$. Therefore, $\text{Arg}(\zeta_m/(z_{k,j} - z_0)) \in (-\pi/2, \pi/2)$. It follows that $\text{Log}(\zeta_m/(z_{k,j} - z_0)) - \text{Log}(\zeta_p/(z_{k,j} - z_0)) = \text{Log}(\zeta_m/\zeta_p)$ for any $m, p = 0, \dots, 3$. Hence we have

$$\begin{aligned} & \text{Log} \frac{\zeta_1}{\zeta_0} + \text{Log} \frac{\zeta_2}{\zeta_1} + \text{Log} \frac{\zeta_3}{\zeta_2} + \text{Log} \frac{\zeta_0}{\zeta_3} = \text{Log} \frac{\zeta_1}{z_{k,j} - z_0} - \text{Log} \frac{\zeta_0}{z_{k,j} - z_0} \\ & + \text{Log} \frac{\zeta_2}{z_{k,j} - z_0} - \text{Log} \frac{\zeta_1}{z_{k,j} - z_0} + \text{Log} \frac{\zeta_3}{z_{k,j} - z_0} - \text{Log} \frac{\zeta_2}{z_{k,j} - z_0} + \text{Log} \frac{\zeta_0}{z_{k,j} - z_0} - \text{Log} \frac{\zeta_3}{z_{k,j} - z_0} = 0 \end{aligned}$$

This proves (2).

Using the equality $\text{Log} \zeta = -\text{Log}(1/\zeta)$ for any $\zeta \in \mathbb{C} \setminus (-\infty, 0]$, from (2) we get:

$$\begin{aligned} \text{Log} \frac{z_{k+1,j+1} - z_0}{z_{k,j+1} - z_0} - \text{Log} \frac{z_{k+1,j} - z_0}{z_{k,j} - z_0} &= -\text{Log} \frac{z_{k,j+1} - z_0}{z_{k,j} - z_0} - \text{Log} \frac{z_{k+1,j} - z_0}{z_{k+1,j+1} - z_0} \\ &= \text{Log} \frac{z_{k+1,j+1} - z_0}{z_{k+1,j} - z_0} - \text{Log} \frac{z_{k,j+1} - z_0}{z_{k,j} - z_0} \end{aligned}$$

With the help of the latter equality, from (1) we deduce, using the telescopic-sum argument, that for any $j = 0, \dots, N-1$,

$$\begin{aligned} \psi_{j+1}(1) - \psi_{j+1}(0) - (\psi_j(1) - \psi_j(0)) &= \sum_{k=0}^{N-1} \left(\text{Log} \frac{z_{k+1,j+1} - z_0}{z_{k+1,j} - z_0} - \text{Log} \frac{z_{k,j+1} - z_0}{z_{k,j} - z_0} \right) \\ &= \text{Log} \frac{z_{N,j+1} - z_0}{z_{N,j} - z_0} - \text{Log} \frac{z_{0,j+1} - z_0}{z_{0,j} - z_0} = \text{Log} \frac{\gamma(1, \theta_{j+1}) - z_0}{\gamma(1, \theta_j) - z_0} - \text{Log} \frac{\gamma(0, \theta_{j+1}) - z_0}{\gamma(0, \theta_j) - z_0} = 0. \end{aligned}$$

The last equality holds because $\gamma(1, \theta) = \gamma(0, \theta)$ by the definition of a homotopical family. This completes the proof. $\square$