Solutions, exam TMA 4180

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Problem 1

a) The function f is a sum of two non-negative functions. Iff $x=y=0$ then $f(x,y)=0$. Therefore $x=y=0$ is the point of global minimum.

b) The function f is twice continuously differentiable everywhere with the Hessian

$$\nabla^2 f = \begin{pmatrix} 4 & -12 \\ -12 & 36+12y^2 \end{pmatrix}$$

The principal minors are 4 ~~and~~ $36+12y^2$, and $4(36+14y^2)-144 = 56y^2$

which are all non-negative $\forall x,y \in \mathbb{R}$

Therefore $\nabla^2 f$ is positive semi-definite

~~and~~ f is convex.

c) $f(3,1) = 2(3-3)^2 + 1^4 = 1 =: f_0$

$$\nabla f(3,1) = \begin{pmatrix} 4x - 12y \\ -12x + 36y + 4y^3 \end{pmatrix}\Bigg|_{\substack{x=3 \\ y=1}}$$

$$= \begin{pmatrix} 12-12 \\ -36+36+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Steepest descent direction: $p = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$; $\nabla f^T \cdot p = 16$

Armijo linesearch:

- $\alpha = 1$, $f(3+1\cdot 0, 1-1\cdot 4) = 369 > f_0 - \alpha \cdot c \cdot \nabla f \cdot p$
  - rejected
- $\alpha = 0.1$, $f(3, 1-0.1\cdot 4) \approx 3.0096 > f_0 - \alpha \cdot c \cdot \nabla f \cdot p$
  - rejected
- $\alpha = 0.01$, $f(3, 1-0.01\cdot 4) \approx 0.87815$

  $f_0 - \alpha \cdot c \cdot \nabla f \cdot p = 1 - 0.01 \cdot 0.25 \cdot 16 = 0.96$

  - accepted!

$$\begin{pmatrix} x \\ y \end{pmatrix}^{new} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha \cdot p = \begin{pmatrix} 3 \\ 1-0.01\cdot 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0.96 \end{pmatrix}$$

Problem 2

a) Take any $x_1, x_2 \in \mathbb{R}^n$, $0 \le \lambda \le 1$

Then $f(x_1) \le \max\{f(x_1), f(x_2)\}$

$\Rightarrow \lambda f(x_1) \le \lambda \max\{f(x_1), f(x_2)\}$ $(\lambda \ge 0)$

Similarly

$(1-\lambda) f(x_2) \le (1-\lambda) \max\{f(x_1), f(x_2)\}$ $(1-\lambda \ge 0)$

$\Rightarrow$

$f(\lambda x_1 + (1-\lambda) x_2) \le$ (convexity of f)

$\lambda f(x_1) + (1-\lambda) f(x_2) \le$

$\underbrace{[\lambda + (1-\lambda)]}_{=1} \max\{f(x_1), f(x_2)\}$

$\Rightarrow$ f is quasi-convex

b) Suppose that f is quasi-convex,

$\alpha \in \mathbb{R}$ - arbitrary, $x_1, x_2 \in S_\alpha$,

$0 \le \lambda \le 1$ - arbitrary.

Need to show: $\lambda x_1 + (1-\lambda) x_2 \in S_\alpha$.

$$f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\} \le \alpha$$

quasi-convexity

because $x_1 \in S_\alpha \Leftrightarrow f(x_1) \le \alpha$

$x_2 \in S_\alpha \Leftrightarrow f(x_2) \le \alpha$

$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S_\alpha \Rightarrow S_\alpha$ - convex set

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Suppose now that $\forall \alpha \in \mathbb{R}$, S_α is a convex set.

Take any $x_1, x_2 \in \mathbb{R}^n$, and put $\alpha = \max\{f(x_1), f(x_2)\}$

Then $\forall\ 0 < \lambda \le 1$:

$\lambda x_1 + (1-\lambda)x_2 \in S_\alpha$, since

$x_1 \in S_\alpha$, $x_2 \in S_\alpha$, and S_α is a convex set.

But $\lambda x_1 + (1-\lambda)x_2 \in S_\alpha \Leftrightarrow$

$\lambda x_1 + (1-\lambda) x_2 < \alpha = \max\{f(x_1), f(x_2)\}$

$\Rightarrow f$ - quasi-convex function.

c) f is quasi-convex, sine its level-sets are convex (by b)

$$S_\alpha = \begin{cases} \emptyset, & \alpha < 0 \\ [0,1], & 0 \le \alpha < 1 \\ \mathbb{R}, & 1 \le \alpha \end{cases}$$

f is not convex, since e.g.

$$1 = f(2) \not\le \tfrac{1}{2} f(0) + \tfrac{1}{2} f(4) = 0 + \tfrac{1}{2} = \tfrac{1}{2}.$$

Also, every point in $\mathbb{R}$ is a point of local minimum, but only points in $[0,1]$ are points of global minimum.

Problem 3

We need to show that all search directions are steepest descent directions, ~~or~~ that is, that $\beta_{k+1} = 0$.

$$0 = p_{k+1}^T p_k = -\nabla\varphi(x_{k+1})^T p_k + \beta_{k+1} \underbrace{p_k p_k}_{\geq 0} \qquad (*)$$

x_{k+1} is calculated using the exact linesearch along p_k. That is

$$x_{k+1} = \arg\min_{\alpha} \varphi(x_k + \alpha p_k)$$

Since φ is continuously differentiable

$$\Rightarrow \quad \frac{d}{d\alpha}\varphi(x_k + \alpha p_k)\Big|_{\alpha=\alpha_k} = 0$$

$$\nabla\varphi(\underbrace{x_k + \alpha_k p_k}_{x_{k+1}})^T p_k = 0$$

$\Rightarrow$ substitute into $(*)$ $\Rightarrow$ $\beta_{k+1}\|p_k\|^2 = 0$

$\beta_{k+1} = 0.$

$\blacksquare$

Problem 4

a) Both constraints are active at $(3,4)$.

Their gradients are $\begin{pmatrix} -6 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

– linearly independent $\left(\text{e.g. } \begin{vmatrix} -6 & 4 \\ -8 & -3 \end{vmatrix} = 18+24 \neq 0 \right)$

Therefore LICQ is satisfied and both cones coincide.

The cone of linearized feasible directions is, per definition

$$T = \left\{ d_1, d_2 : \begin{array}{l} -6d_1 - 8d_2 \geq 0 \\ 4d_1 - 3d_2 \geq 0 \end{array} \right\}$$

b) Both constraints are concave (one is affine, the other one has a ~~...~~ non-positively definite Hessian $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$).

The objective function is convex (affine).

LICQ hold $\Rightarrow$ KKT-conditions are necessary and sufficient for optimality.

KKT: $$\left.\begin{pmatrix} 1 \\ \pi \end{pmatrix} - \lambda_1 \begin{pmatrix} -2x \\ -2y \end{pmatrix} - \lambda_2 \begin{pmatrix} 4 \\ -3 \end{pmatrix}\right|_{\substack{x=3 \\ y=4}} = 0$$

$$\lambda_1, \lambda_2 \geq 0$$

(Both constraints are active, so complementarity slackness is satisfied)

$$\begin{cases} 6\lambda_1 - 4\lambda_2 = -1 \\ 8\lambda_1 + 3\lambda_2 = -\pi \end{cases}$$

$$\lambda_2 = \frac{6\lambda_1 + 1}{4}$$

$$8\lambda_1 + \frac{18\lambda_1}{4} = -\pi - \frac{3}{4}$$

$$\lambda_1 = \left(-\pi - 3/4\right) \cdot \frac{2}{25}$$

$$\lambda_1 \geq 0 \iff -\Pi - 3/4 \geq 0 \iff \underline{\underline{\Pi \leq -3/4}}$$

$$\lambda_2 = \frac{6}{4} \cdot \frac{2}{25}\left(-\Pi - \frac{3}{4}\right) + \frac{1}{4} \geq 0$$

$$\underbrace{-\Pi - \frac{3}{4}}_{\geq 0} \geq -\frac{1}{4} \cdot \frac{50}{6}$$

– satisfied $\underline{\underline{\forall \Pi \leq -3/4}}$

Answer: $\Pi \leq -3/4$

Problem 5

a) From the constraint: $x = y+1$

$$\min_{(x,y)\in\Omega} f(x,y) \Leftrightarrow \min_{y} \frac{1}{2}\left(y^2 + (y+1)^2\right) =: g(y)$$

$g(y) = y^2 + y + \frac{1}{2}$ - C^1, convex (optimality conditions are necessary & sufficient)

$g' = 2y + 1 = 0 \qquad \underline{\underline{y^* = -\frac{1}{2} \qquad x^* = \frac{1}{2}}}$

$$\left.\nabla f - \lambda^* \nabla c\right|_{(x,y)=(x^*,y^*)} = 0$$

$$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} - \lambda^* \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \Rightarrow \lambda^* = \frac{1}{2}.$$

b) We combine the objective & the constraint:

$$\min_{(x,y)\in\mathbb{R}^2} f + \frac{\mu}{2}c^2 = \frac{1}{2}(x^2+y^2) + \frac{\mu}{2}(x-y-1)^2$$

$$= \frac{\mu+1}{2}x^2 + \frac{\mu+1}{2}y^2 - \mu xy - \mu x + \mu y + \frac{\mu}{2}$$

This is a convex unconstrained minimization problem; minimum is attained when the gradient vanishes.

$$\begin{cases} (\mu+1)x - \mu y = \mu \\ -\mu x + (\mu+1)y = -\mu \end{cases}$$

$$y = \frac{\mu+1}{\mu}x - 1$$

$$-\mu x + \frac{(\mu+1)^2}{\mu}x = -\mu + \mu + 1$$

$$\frac{2\mu+1}{\mu}x = 1 \Rightarrow \boxed{x = \frac{\mu}{2\mu+1} \quad y = \frac{\mu+1}{2\mu+1} - 1 = -\frac{\mu}{2\mu+1}}$$

If $\mu = 2 \Rightarrow x = \frac{2}{5}, y = -\frac{2}{5}$.

c) Augmented Lagrangian:

$$L_A = f + \lambda \cdot c + \frac{\mu}{2} c^2$$

$$= \frac{\mu+1}{2} x^2 + \frac{\mu+1}{2} y^2 - \mu x y - (\mu+\lambda) x + (\mu+\lambda) y$$

$$+ \frac{\mu}{2} + \lambda$$

- Convex smooth function of x, y.

$$\nabla_{x,y} L_A = 0 \iff$$

$$\begin{cases} (\mu+1) x - \mu y = \mu + \lambda \\ -\mu x + (\mu+1) x = -(\mu+\lambda) \end{cases}$$

$$y = \frac{\mu+1}{\mu} x - \frac{\mu+\lambda}{\mu}$$

$$-\mu x + \frac{(\mu+1)^2}{\mu} x = -(\mu+\lambda) + \frac{(\mu+1)(\mu+\lambda)}{\mu}$$

$$\frac{2\mu+1}{\mu} x = (\mu+\lambda)\left[\frac{\mu+1}{\mu} - 1\right] = \frac{\mu+\lambda}{\mu}$$

$$x = \frac{\mu+\lambda}{2\mu+1}$$

$$y = \frac{(\mu+1)(\mu+\lambda)}{\mu(2\mu+1)} - \frac{\mu+\lambda}{\mu} = -\frac{\mu+\lambda}{2\mu+1}$$

If $\mu = 2$, $\lambda = \frac{1}{2}$

$$\Rightarrow \left.\begin{aligned} x &= \frac{5/2}{5} = \frac{1}{2} \\ y &= -\frac{1}{2} \end{aligned}\right\} \text{ - exact solution to the original problem.}$$