TMA 4180 Optimeringsteori
Exam May 15, 2004
Solution with additional comments

The problems are included *only* for easing the reading!

Problem 1

Consider the unconstrained minimization problem

$$\min_{x=(\xi,\eta)\in\mathbb{R}^2} f(\xi,\eta) = \min_{x=(\xi,\eta)\in\mathbb{R}^2} \left\{5 - 2\eta - 4\xi + 2\xi\eta + \xi^2 + 2\eta^2\right\}. \tag{1}$$

(a) Compute the gradient and the Hessian of f in an arbitrary point, and show that $x^* = (3,-1)'$ is the unique global minimum.

(b) Start at $x_0 = (0,0)'$ and verify that one iteration with the Steepest Descent method brings you to $x_1 = (1, 1/2)'$.

(c) Explain the Conjugate Gradient (CG) method applied to the quadratic model problem

$$Q(x) = \frac{1}{2}x'Ax - b'x, \tag{2}$$

and show that if we start the CG method in $x_0 = 0$ with $d_0 = -\nabla Q(0) = b$ as the first basis vector, then x_1 is identical to the first iteration with the Steepest Descent method starting from 0.

(d) Starting from x_1 in (b), state (without any computations) the result of the next iteration with the CG method applied to the problem in (1), and verify that the corresponding search directions for the two CG iterations are (conjugate) orthogonal with respect to the Hessian of f.

Solution:

(a) The gradient and the Hessian are given by

$$\begin{aligned} \nabla f(\xi,\eta) &= (-4 + 2\eta + 2\xi, -2 + 2\xi + 4\eta), \\ \nabla^2 f &= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, \end{aligned} \tag{3}$$

and we note that $\nabla^2 f$ is *positive definite* everywhere. The function f is hence *strictly convex* for all $x \in \mathbb{R}^2$. By putting $(3,-1)'$ into the expression for ∇f, we see that $\nabla f(3,-1) = 0$, and since f is strictly convex, this will be a *unique global minimum.*

Some manipulations actually show that

$$\begin{aligned} f(\xi,\eta) &= f(0,0) + \nabla f(0,0)\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \frac{1}{2}(\xi,\eta)\begin{pmatrix}2 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \\ &= 5 + (-4,-2)\begin{pmatrix}\xi\\ \eta\end{pmatrix} + (\xi,\eta)\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \\ &= (\xi-3,\eta+1)\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\xi-3\\ \eta+1\end{pmatrix}. \end{aligned} \tag{4}$$

(b) The gradient at $(0,0)'$ is $(-4,-2)$, and x_1 follows simply by finding the minimum of $f(x)$ along the *ray*

$$x(\alpha) = (0,0)' + \alpha(4,2)' = \alpha(4,2)', \ \alpha \geq 0.$$

We thus compute

$$\frac{d}{d\alpha}\left(5 - 2(2\alpha) - 4(4\alpha) + 2(2\alpha)(4\alpha) + (4\alpha)^2 + 2(2\alpha)^2\right) = 80\alpha - 20 = 0,$$

that is, $\alpha = \frac{1}{4}$, and

$$x_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}. \tag{5}$$

It is equally simple to compute ∇f along the ray and find α from the requirement that

$$\nabla f(\alpha(4,2)) \cdot (4,2)' = 0. \tag{6}$$

(c) The main idea behind the CG method for the N dimensional model problem is to find an expansion of the solution x^* of the form

$$x^* = \sum_{n=0}^{N-1} \alpha_n d_n, \tag{7}$$

where $\{d_n\}_{n=0}^{N-1}$ is an A-orthogonal basis, that is

$$d_n' A d_m = 0 \tag{8}$$

for $m \neq n$. By utilizing that $Ax^* = b$, it is not even necessary to know x^* in order to determine $\{\alpha_n\}$:

$$\alpha_n = \frac{\langle d_n, x^* \rangle_A}{\langle d_n, d_n \rangle_A} = \frac{d_n' A x^*}{d_n' A d_n} = \frac{d_n' b}{d_n' A d_n}. \tag{9}$$

The basis vectors $\{d_n\}$ are found iteratively and are all of the form

$$d_n = -\nabla Q(x_n)' + \beta_{n-1} d_{n-1}$$

(a somewhat surprising fact!).

Given $d_0 = b$, α_0 is determined by

$$\alpha_0 = \frac{d_0' A x^*}{d_0' A d_0} = \frac{b' b}{b' A b}. \tag{10}$$

For the SD method, we determine how far we have to go along the b direction by the equation

$$\nabla Q(\alpha_0 b) \cdot b = (A(\alpha_0 b) - b)' b = 0, \tag{11}$$

which leads to the same answer for α_0.

(d) Since the problem in (**??**) is 2-dimensional, the CG method will converge in at most 2 iterations, and if x_1 is the first iteration, the next will actually bring us to the solution,

$$x_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}. \tag{12}$$

It remains to be proved that $d_1 \propto x_1$ and $d_2 \propto x_2 - x_1$ are A-orthgonal:

$$x_1' A (x_2 - x_1) \propto \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 - 1 \\ -1 - \frac{1}{2} \end{pmatrix} = 0. \tag{13}$$

Problem 2:

(a) State the Karush-Kuhn-Tucker Theorem for a local minimum x^* of a function $f(x)$ subject to sets of equality, $\{c_i(x) = 0,\ i \in \mathcal{E}\}$, and inequality, $\{c_i(x) \geq 0,\ i \in \mathcal{I}\}$, constraints.

In the rest of this problem we consider an inequality constrained optimization problem

$$\min_{x \in \Omega} f(x), \tag{14}$$

$$\Omega = \{x\ ;\ c_i(x) \geq 0,\ i \in \mathcal{I}\}, \tag{15}$$

where the objective function $f(x)$ and $-c_i(x)$ are convex functions for all $i \in \mathcal{I}$.

(b) Prove that Ω is a convex set.

(c) Assume that x^* is a KKT-point, that is,

$$\begin{aligned} \nabla \mathcal{L}(x^*, \lambda^*) &= 0, \\ \lambda_i^* \cdot c_i(x^*) &= 0,\ i \in \mathcal{I} \\ x^* \in \Omega\ ,\ \lambda^* &\geq 0, \end{aligned} \tag{16}$$

where $\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{I}} \lambda_i c_i(x)$.

Prove that x^* is a global minimum for the problem defined in Eqns. (14) and (15).

(d) Consider

$$\begin{aligned} f(x, y) &= x + 2y, \\ c_1(x, y, z) &= y \geq 0, \\ c_2(x, y, z) &= 2 - (x - 2)^2 - y^2 \geq 0, \\ c_3(x, y, z) &= 1 - x^2 - y^2 \geq 0. \end{aligned} \tag{17}$$

and explain why this is a problem of the form above. Make a sketch and guess a solution. Then show that (c) is fulfilled for the point you have found.

Solution:

(a) The Karush-Kuhn-Tucker Theorem (KKT Theorem) in our formulation considers the following problem:

$$\begin{aligned} &\min_x f(x), \\ &c_i(x) = 0,\ i \in \mathcal{E}, \\ &c_i(x) \geq 0,\ i \in \mathcal{I}. \end{aligned} \tag{18}$$

Active constraints ($\mathcal{A}$) in a point x are all equality constraints and the subset among the inequality constraints where $c_i(x) = 0$. The *Linear Independence Constraint Qualification* (LICQ) holds at x if $\{\nabla c_i(x)\}_{x\in\mathcal{A}}$ are linearly independent.

The KKT Theorem: *Assume that x^* is a local minimum for 18 and that the LICQ holds in x^*.Then there is a vector of Lagrange multipliers, λ^*, such that*

$$\begin{array}{rc} & \nabla f(x^*) = \sum_{i\in\mathcal{I}\cup\mathcal{E}} \lambda_i^* \nabla c_i(x^*), \\ (i) & \lambda_i^* \cdot c_i(x^*) = 0, \ i \in \mathcal{E}\cup\mathcal{I}, \\ (ii) & \lambda_i^* \geq 0 \text{ for } i \in \mathcal{I}. \end{array} \tag{19}$$

(b) We first recall that *for all convex functions, $\phi_i(x)$, the sets*

$$\Omega_i) = \{x; \phi_i(x) \leq c\ \} \tag{20}$$

are convex. The intersection of convex sets is convex, and hence for a collection of convex functions, the set

$$\{x; \phi_i(x) \leq c\ , i = 1, \cdots, n\} \tag{21}$$

will be convex. Now,

$$\Omega = \{x; c_i(x) \geq 0\ , i \in \mathcal{I}\} = \{x; -c_i(x) \leq 0\ , i \in \mathcal{I}\}, \tag{22}$$

which is then convex.

(c) Since $\lambda^* \geq 0$, the Lagrange function

$$\begin{aligned} \mathcal{L}(x, \lambda^*) &= f(x) - \sum_{i\in\mathcal{I}} \lambda_i^* c_i(x) \\ &= f(x) + \sum_{i\in\mathcal{A}} \lambda_i^* (-c_i(x)) \end{aligned} \tag{23}$$

will be convex in x (NB! $\lambda_i^* = 0$ for all $i \notin \mathcal{A}$). But x^* will then, since $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$, be an unconstrained *global minimum* for $\mathcal{L}(x, \lambda^*)$. Thus, we have for all $x \in \Omega$, since $\lambda_i^* c_i(x) \geq 0$ for all i,

$$\begin{aligned} f(x) &\geq f(x) - \sum_{i\in\mathcal{A}} \lambda_i^* c_i(x) \\ &= \mathcal{L}(x, \lambda^*) \\ &\geq \mathcal{L}(x^*, \lambda^*) \\ &= f(x^*) - \sum_{i\in\mathcal{A}} \lambda_i^* c_i(x^*) \\ &= f(x^*), \end{aligned} \tag{24}$$

since $c_i(x^*) = 0$ for all $i \in \mathcal{A}$.

(d) First of all, f and c_1 are linear and hence convex regardless of signs. Moreover, both $-c_2$ and $-c_3$ have positive definite Hessians.

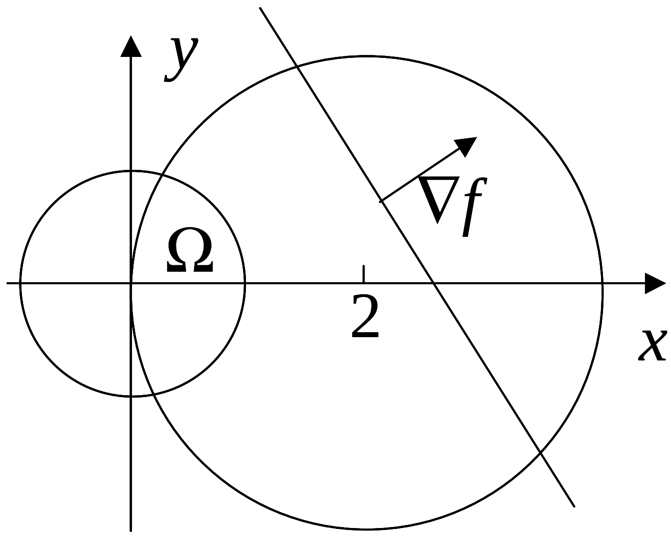

Figure 1: Constraints and the feasible domain for the problem in 2 (d).

The domain in the $x - y$-plane and ∇f is sketched in Fig. 1. From the gradient of f it is obvious that the solution is $x^* = (0,0)'$. All constraints are fulfilled, $c_1$and c_2 are active, and

$$\begin{aligned}\nabla c_1(0,0) &= \mathbf{j},\\ \nabla c_2(0,0) &= 4\mathbf{i},\end{aligned} \tag{25}$$

Thus,

$$\nabla f(0,0) = \mathbf{i} + 2\mathbf{j} = \frac{1}{4}\nabla c_2(0,0) + 2\nabla c_1(0,0), \tag{26}$$

and

$$\lambda^* = \begin{pmatrix} 2 \\ 1/4 \end{pmatrix} > 0. \tag{27}$$

Problem 3:

(a) Explain what is meant by the standard form of a Linear Programming (LP) problem. Transform the following problem to the standard form:

$$\begin{aligned} &\max_{x_1,x_2}\{2x_2 + x_1\},\\ &x_1 \leq 4 + x_2, &(28)\\ &x_2 \geq 1 - 4x_1,\\ &x_2 \geq 0. &(29)\end{aligned}$$

(b) Determine the minimum value of the objective function in the following problem:

$$\begin{aligned} &\min\{7x_1 + 2x_2 + 3x_3 + x_4 + 2x_5\},\\ &5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 = 1, &(30)\\ &x_i \geq 0\end{aligned}$$

Hint: The dual problem of

$$\begin{aligned} &\min c'x, \\ &Ax = b, \\ &x \geq 0, \end{aligned} \tag{31}$$

is

$$\begin{aligned} &\max_{\lambda} b'\lambda, \\ &A'\lambda \leq c. \end{aligned} \tag{32}$$

Apply the KKT-equations for the dual problem.

Solution:

(a) The standard form is as follows:

$$\begin{aligned} &\min_{x} c'x \\ &Ax = b, \\ &x \geq 0. \end{aligned}$$

Usually, it is also added that A should have full row rank.

In our problem, there is no bound on x_1, so we write $x_1 = y_1 - y_2$, $y_1,y_2 \geq 0$, and introduce two additional slack variables s_1 and s_2 in order to convert the inequalities to equalities:

$$\begin{aligned} &\min_{y_1,y_2,x_2} \{-2x_2 - y_1 + y_2\}, \\ &y_1 - y_2 + s_1 = 4 + x_2, \\ &x_2 = s_2 + 1 - 4(y_1 - y_2). \end{aligned} \tag{33}$$

This can be written as

$$\min (-2, -1, 1) \cdot (x_2, y_1, y_2)' \tag{34}$$

$$\begin{aligned} s_1 + y_1 - y_2 - x_2 &= 4, \\ s_2 - 4y_1 + 4y_2 - x_2 &= -1, \\ s_1, s_2, y_1, y_2, x_2 &\geq 0, \end{aligned} \tag{35}$$

from which b and A follows at once.

(b) In the present case,

$$c' = (7, 2, 3, 1, 2), \tag{36}$$

and

$$\begin{aligned} A &= [5, 4, 3, 2, 1], \\ b &= (1). \end{aligned}$$

The dual problem becomes

$$\begin{aligned}
\max_{\lambda} \lambda \cdot 1 = \max_{\lambda} \lambda, \\
5\lambda \leq 7, \\
4\lambda \leq 2, \\
3\lambda \leq 3, \\
2\lambda \leq 1, \\
1\lambda \leq 2,
\end{aligned} \tag{37}$$

which clearly has the solution $\lambda^* = 1/2$. By the Duality Theorem, we then also know the optimal value of the objective function,

$$\min c'x = \max \lambda = 1/2. \tag{38}$$

In order to find the x-es, we let x be the Lagrange multipliers for the dual problem so that

$$\mathcal{L}(\lambda, x) = (-b)' \lambda - x' (c - A'\lambda)$$

(NB! Remember to turn $\max b'\lambda$ to $\min (-b)' \lambda$ before stating the KKT equations!). The KKT equations are then

$$\begin{aligned}
\nabla_\lambda \mathcal{L}(\lambda, x) &= -b + Ax = 0, \\
x_i (c - A'\lambda)_i &= 0, \ \ i = 1, 2, \cdots, n, \\
x &\geq 0.
\end{aligned} \tag{39}$$

Of course, the KKT-equations are the same for the dual and the primal problems, but the above equations are in a form that can be used right away.

We know that $\lambda^* = 1/2$, so that

$$c - A'\lambda^* = \begin{pmatrix} 7 - 5 \cdot \frac{1}{2} \\ 2 - 4 \cdot \frac{1}{2} \\ 3 - 3 \cdot \frac{1}{2} \\ 1 - 2 \cdot \frac{1}{2} \\ 2 - 1 \cdot \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ 0 \\ \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} \geq 0. \tag{40}$$

Since $x_i (c - A'\lambda)_i = 0$, the only components of x that are non-zero are x_2 and x_4, and the KKT equations for the convex problem (*cf.* Problem 2!) above are satisfied for

$$\begin{aligned}
x_1 = x_3 = x_5 = 0 \\
4x_2 + 2x_4 = 1 \ , \ \ x_2, x_4 \geq 0.
\end{aligned} \tag{41}$$

All these solutions are thus global minima with $\min c'x = 1/2$ (which is easily verified!).

Problem 4

(a) Define what is meant by a convex and a strictly convex functional $J(y)$ defined for all y-s in a convex domain D of functions, and show that all functions $y_0 \in D$ such that $\delta J(y_0; v) = 0$ are global minima.

Many control problems lead to the minimization of a functional of the form

$$J(y) = \int_0^T \left(y(t)^2 + \dot{y}(t)^2\right) dt, \; y \in C^1[0,T] \tag{42}$$

(b) Show that J is strictly convex.

(c) Let

$$\mathcal{D} = \left\{ \; y \in C^1[0,T] \; ; \; y(0) = 1 \; , \; y(T) \text{ is free} \right\}. \tag{43}$$

Solve the optimization problem

$$\min_{y \in \mathcal{D}} J(y) \tag{44}$$

when

$$G(y) = \int_0^T y(t)\, dt = 0. \tag{45}$$

(d) Consider the functional

$$H(y) = \left(\int_0^T y(t)\, dt\right)^2, \; y \in C[0,T] \tag{46}$$

Show that the functional is convex, but not strictly convex.

(e) Solve the problem

$$\min_{y \in \mathcal{D}} \{J(y) + \mu H(y)\} \; , \; \mu > 0,$$
$$\mathcal{D} = \left\{ \; y \in C^1[0,T] \; ; \; y(0) = 1 \; , \; y(T) \text{ is free} \right\} \tag{47}$$

What happen to the solutions when $\mu \to \infty$?

Hint: Use partial integration to get rid of v' in $\delta J(y;v)$, and recall the equation in (c).

Solution:

(a) The functional J is *convex* if

$$J(y+v) - J(y) \geq \delta J(y;v) \tag{48}$$

for all $y \; , \; y+v \in \mathcal{D}$. It is *strictly convex* if the inequality is sharp for all $v \neq 0$.

If $\delta J(y_0; v) = 0$,

$$J(y) - J(y_0) = J(y_0 + y - y_0) - J(y_0) \geq \delta J(y_0; y - y_0) = 0. \tag{49}$$

Hence, $J(y) \geq J(y_0)$ for all $y \in \mathcal{D}$, and y_0 is a global minimum.

(b) This follows immediately since the integrand is *strongly convex*. It can also be seen directly if we use that

$$\delta J(y;v) = 2\int_0^T [yv + \dot{y}\dot{v}]\, dt, \tag{50}$$

since

$$J(y+v)-J(y)=\int_0^T\left[(y+v)^2+(\dot{y}+\dot{v})^2-y^2-\dot{y}^2\right]dt$$
$$=\delta J(y;v)+\int_0^T\left[v^2+\dot{v}^2\right]dt. \tag{51}$$

The last integral is strictly positive for all non-zero functions $v\in C^1[0,T]$.

(c) We introduce a Lagrange multiplier λ and consider the extended functional

$$\mathcal{L}(y,\lambda)=\int_0^T\left(y(t)^2+\dot{y}(t)^2\right)dt+\lambda\int_0^T y(t)\,dt,$$
$$=\int_0^T\left(y(t)^2+\dot{y}(t)^2+\lambda y(t)\right)dt. \tag{52}$$

The Euler equation becomes

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)-\frac{\partial f}{\partial y}=\frac{d}{dx}(2\dot{y})-\lambda-2y=0, \tag{53}$$

or

$$\ddot{y}-y=\frac{\lambda}{2}, \tag{54}$$

with the general solution

$$y(t)=A\cosh t+B\sinh t-\frac{\lambda}{2}. \tag{55}$$

The boundary conditions are a fixed end-point condition at $t=0$ and a natural condition at $t=T$:

$$y(0)=1,$$
$$\left(\frac{\partial f}{\partial \dot{y}}\right)(T)=2\dot{y}(T)=0, \tag{56}$$

and hence,

$$y(t)=\left(1+\frac{\lambda}{2}\right)\frac{\cosh(T-t)}{\cosh T}-\frac{\lambda}{2}. \tag{57}$$

We determine λ by requiring that $\int_0^T y(t)\,dt=0$, that is,

$$\int_0^T\left(\left(1+\frac{\lambda}{2}\right)\frac{\cosh(T-t)}{\cosh T}-\frac{\lambda}{2}\right)dt=\left(1+\frac{\lambda}{2}\right)\frac{\sinh T}{\cosh T}-\frac{\lambda T}{2}=0,$$

or

$$\lambda=\frac{2\tanh T}{T-\tanh T}. \tag{58}$$

Hence,

$$y^*(t)=\left(1+\frac{\tanh T}{T-\tanh T}\right)\frac{\cosh(T-t)}{\cosh T}-\frac{\tanh T}{T-\tanh T}. \tag{59}$$
$$=\frac{T\frac{\cosh(T-t)}{\cosh T}-\tanh T}{T-\tanh T}=\frac{T\cosh(T-t)-\sinh T}{T\cosh T-\sinh T}. \tag{60}$$

(A unique solution since $J(y) + \lambda \int_0^T y(t)\,dt$ is strictly convex and the solution for λ is unique).

(d) We need to compute the derivative, and use the standard formula

$$\delta H(y; v) = \left.\frac{dH(y + \varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0} = 2\left(\int_0^T y(t)\,dt\right)\int_0^T v(t)\,dt. \tag{61}$$

Now,

$$\begin{aligned} H(y+v) - H(y) &= \left(\int_0^T y(t)\,dt + \int_0^T v(t)\,dt\right)^2 - \left(\int_0^T y(t)\,dt\right)^2 \\ &= 2\left(\int_0^T y(t)\,dt\right)\int_0^T v(t)\,dt + \left(\int_0^T v(t)\,dt\right)^2 \\ &\geq 2\left(\int_0^T y(t)\,dt\right)\int_0^T v(t)\,dt = \delta H(y; v), \end{aligned} \tag{62}$$

which shows that H is convex. However,

$$H(y+v) - H(y) = 0 \tag{63}$$

for all v such that $\int_0^T v(t)\,dt = 0$, so that H is *not strictly convex.*

(e) We follow the hint, and re-derive the Euler equation:

$$\begin{aligned} \delta J(y; v) &= \int_0^T [2yv + 2\dot{y}\dot{v}]\,dt \\ &= [2\dot{y}v]_0^T + \int_0^T \left[-\frac{d}{dx}(2\dot{y}) + 2y\right] v(t)\,dt. \end{aligned} \tag{64}$$

Thus,

$$\begin{aligned} \delta(J(y,v) + \mu H(y;v)) &= [2\dot{y}v]_0^T + \int_0^T \left[-\frac{d}{dx}(2\dot{y}) + 2y\right] v(t)\,dt + 2\mu\left(\int_0^T y(t)\,dt\right)\int_0^T v(t)\,dt \\ &= [2\dot{y}v]_0^T + 2\int_0^T \left[-\ddot{y} + y + \mu\int_0^T y(s)\,ds\right] v(t)\,dt. \end{aligned} \tag{65}$$

Since the functional $J(y,v) + \mu H(y;v)$ is convex, it is enough to ensure that

$$\delta[J(y,v) + \mu H(y;v)] = 0 \tag{66}$$

for all allowed v-s. This is fulfilled if

$$\begin{aligned} -\ddot{y} + y + \mu\int_0^T y(s)\,ds &= 0, \\ y(0) &= 1, \\ \dot{y}(T) &= 0. \end{aligned} \tag{67}$$

The first equation looks strange, but if we let the integral be equal to a constant, say

$$\int_0^T y(s)\,ds = C, \tag{68}$$

we get exactly the same problem as in (c). The solution fulfilling $\dot{y}(T) = 0$ is

$$y(t) = A\cosh(T-t) - \mu C, \tag{69}$$

and A and C are determined from

$$\begin{aligned} y(0) = 1 &= A\cosh T - \mu C, \\ C &= \int_0^T y(s)\,ds = A\sinh T - \mu C T, \end{aligned} \tag{70}$$

or

$$\begin{aligned} C &= \frac{\sinh T}{1+\mu T} A, \\ A &= \frac{-(T\mu+1)}{\mu\sinh T - (T\mu+1)\cosh T}. \end{aligned}$$

The solution is unique, since $J + \mu H$ is strictly convex.
When $\mu \to \infty$,

$$\begin{aligned} \lim_{\mu\to\infty} A &= \frac{-T}{\sinh T - T\cosh T}, \\ \lim_{\mu\to\infty} \mu C &= \frac{\sinh T}{\sinh T - T\cosh T}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} \lim_{\mu\to\infty} y^*(t) &= \frac{-T}{\sinh T - T\cosh T}\cosh(T-t) + \frac{\sinh T}{\sinh T - T\cosh T} \\ &= \frac{T\cosh(T-t) - \sinh T}{T\cosh(T) - \sinh T}, \end{aligned} \tag{72}$$

which we recognize as the solution in (c)!