Norwegian University of Science and Technology
Department of Mathematical Sciences

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Exam in TMA4180 Optimization Theory

Monday June 6, 2011
Tid: 09.00 – 13.00

Auxiliary materials: Simple calculator (Hewlett Packard HP30S or Citizen SR-270X)
Rottmann: *Matematisk formelsamling*

Problem 1

a) Find all minima (in $\mathbb{R}^2$) of the function

$$f(x, y) = x^4 - 2x^2 + 3y^2 - 12y.$$

List all the general results you are using.

b) Formulate the steepest descent method for this problem (but you do not have to perform any iterations).

Estimate the drop in the error per iteration (expressed in terms of the appropriate norm) near a global minimum of the problem given in **a)**.

Problem 2 Consider a differentiable function $f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}$. What do we mean by a convex function? Write down the definition of the tangent plane $T_{x_0}(x)$ of f in a point x_0. Assume that f has the property that $f(x) \geq T_{x_0}(x)$ for all points x_0 and x. Show that the function f is convex.

Problem 3 Given the problem

$$\min\ y - x$$

subject to

$$\begin{aligned} -x + y &\geq 0 \\ x + y &\geq 0 \\ 2x^2 + y - 1 &\geq 0 \\ -y + 2 &\geq 0 \end{aligned}$$

a) State the KKT conditions, and show that $x^\star = -0.5,\ y^\star = 0.5$ is a KKT-point for the problem.

b) Is $(x^\star, y^\star)$ a minimum? Justify your claim.

Problem 4 Given the linear problem:

$$\max\ x_1 + 2x_2$$

subject to

$$\begin{gathered} -1 \leq x_2 - x_1 \leq 1 \\ 0 \leq x_1 \leq 2 \\ 0 \leq x_2 \end{gathered}$$

a) Plot the feasible region, and solve the problem graphically.

b) Bring the LP problem over to standard form, that is

$$\begin{aligned} \min\ c^T x&, \\ Ax = b&, \qquad \text{A has full row rank,} \\ x \geq 0&. \end{aligned}$$

Problem 5 Given the functional

$$F(y) = \int_0^1 \big(y(x)^2 + x^2 y'(x)\big)dx$$

on $\mathcal{D} = \{y \in C[0,1]\ :\ y(0) = 0,\ y(1) = 1\}$.

a) Find the function $y_0(x)$ that minimize F on $\mathcal{D}$.

b) Explain what we mean with a strictly convex functional. Prove that F is strictly convex.

Problem 6 You and some friends are planning to do some kayaking in the fjord after the exam. You will start from Grillstadsfjæra, your friends from Hansbakkfjæra (two popular beaches in Trondheim), and the group will continue from there. So you will have to paddle a little extra, but would not like to waste too much energy on this. The distance in a straight line between Grillstadfjæra and Hansbakkfjæra is l, and you should paddle it on time T. At that time, your friends are already in the water, ready to start, so you do not have to make a stop at Hansbakkfjæra.

Assuming that the energy consumption depends on the velocity $v(t)$ and the acceleration $\dot{v}(t) = dv(t)/dt$, it may be expressed as

$$F(v) = \int_0^T \big(\dot{v}(t) + \alpha v(t)\big)^2 dt$$

where $\alpha > 0$ is the parameter describing the water resistance. The problem is then to minimize F on

$$\mathcal{D} = \{v \in C^1[0,T] \ : \ v(0) = 0\}$$

under the constraint

$$G(v) = \int_0^T v(t)dt = l.$$

Here, l and T are given constants.

a) Set up the Euler-Lagrange equation for this problem, including boundary conditions.

b) Let $\alpha = 1$ and find the velocity $v_0(t)$ that solves the problem above. Is the solution unique? If so, why?

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spring 2011
Solutions

Not yet proofread

Problem 1

a) *First order necessary condition for a minimum* is $\nabla f(x,y) = 0$, Thus

$$\nabla f(x,y) = \begin{pmatrix} 4x^3 - 4x \\ 6y - 12 \end{pmatrix} = 0$$

gives the possible minima

$$(x,y) = (0,2), \qquad (x,y) = (1,2), \qquad (x,y) = (-1,2)$$

The *second order necessary condition for a minimum* is

$$\nabla f(x,y) = 0, \qquad \nabla^2 f(x,y) \geq 0$$

which in our case becomes:

$$\nabla^2 f(x,y) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 6 \end{pmatrix} \geq 0$$

which clearly is not satisfied for $(x,y) = (0,2)$. Finally, *the sufficient condition for a minimum* is $\nabla f(x,y) = 0$ and $\nabla^2 f(x,y) > 0$, which is satisfied for $(x,y) = (1,2)$ and $(x,y) = (-1,2)$.

We have $f(1,2) = f(-1,2) = -13$. Further, $f(x,y) \to +\infty$ as $x, y \to \pm\infty$, and we can conclude that(1,2) and (-1,2) are global minima.

b) One iteration of the steepest descent method for $\min f(x)$, $f : \mathbb{R}^n \to \mathbb{R}$ is in general given by

$$p_k = -\nabla f(x_k), \qquad \alpha_k = \arg\min_{\alpha>0} f(x_k + \alpha p_k), \qquad x_{k+1} = x_k + \alpha_k p_k.$$

In a practical algorithm, α_k is usually some approximation to the scalar minimization problem, satisfying the Wolfe-conditions:

$$f(x_k + \alpha_k p_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T p_k$$
$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq c_2 \nabla f(x_k)^T p_k.$$

for $0 < c_1 < c_2 < 1$. There are also alternative conditions, see Nocedal and Wright.

In our case, we get:

$$p_k = \begin{pmatrix} -4x_k^3 + 4x_k \\ -6y_k + 12 \end{pmatrix}, \qquad \nabla f(x_k)^T p_k = 16x_k^2(x_k^2 - 1)^2 + 36(y_k - 2)^2.$$

If we denote the Hessian near the gloal minima by A, the error estimate is given by

$$\frac{\|x_{k+1} - x^\star\|_A}{\|x_k - x^\star\|_A} \leq \frac{\kappa(A) - 1}{\kappa(A) + 1}.$$

In our case,

$$A = \begin{pmatrix} 8 & 0 \\ 0 & 6 \end{pmatrix}, \qquad \kappa(A) = 8/6 = 4/3$$

so

$$\frac{\|x_{k+1} - x^\star\|_A}{\|x_k - x^\star\|_A} \leq \frac{1}{7}.$$

Problem 2 A function is convex if for all $x, y \in \mathbb{R}^n$ we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y), \qquad \theta \in [0,1]. \tag{1}$$

A tangent plane of f in x_0 is given by

$$T_{x_0}(x) = f(x_0) + \nabla f(x_0)(x - x_0).$$

Given $x, y \in \mathbb{R}^n$, and $x_\theta = \theta x + (1-\theta)y$. Then

$$f(x_\theta) + \nabla f(x_\theta)(x - x_\theta) \leq f(x)$$
$$f(x_\theta) + \nabla f(x_\theta)(y - x_\theta) \leq f(y)$$

Multiply the first inequality by θ, the second by $1-\theta$ and add them together, and what we get is (1).

Problem 3

a) Let $\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^4 \lambda_i c_i(x)$. A point $x^\star$ is a KKT point if $x^\star \in \Omega$ (that is, all the constraints are satisfied), the LICQ condition is satisfied, and

$$\nabla_x \mathcal{L}(x^\star, \lambda^\star) = 0$$
$$\lambda_i^\star c_i(x^\star) = 0, \qquad \forall i \in \mathcal{E} \cup \mathcal{I}$$
$$\lambda_i^\star \geq 0, \qquad \forall i \in \mathcal{I}$$

Our problem is given by

$$\min\ f(x) = y - x$$

subject to

$$\begin{aligned} c_1(x,y) &= & -x + y & \geq 0 \\ c_2(x,y) &= & x + y & \geq 0 \\ c_3(x,y) &= & 2x^2 + y - 1 & \geq 0 \\ c_4(x,y) &= & -y + 2 & \geq 0 \end{aligned}$$

So $\mathcal{E} = \emptyset$, $\mathcal{I} = \{1, 2, 3, 4\}$.

For $x = -0.5$, $y = 0.5$ we get

$$c_1 = 1 \quad c_2 = 0, \quad c_3 = 0, \qquad c_4 = 3/2$$

so $(x, y) \in \Omega$. The constraints c_1 and c_4 are passive constraints, and from the second KKT condition, this implies that $\lambda_1 = \lambda_4 = 0$. Further, the active constraints satisfies

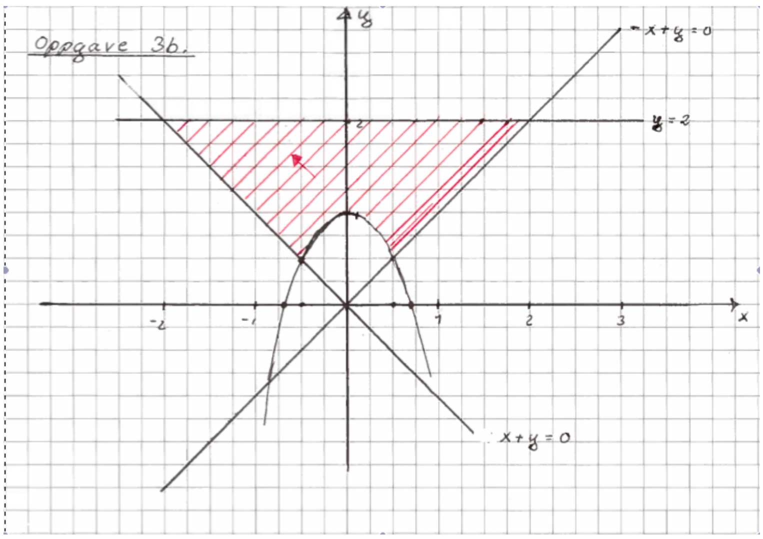
$$\nabla c_2(x,y) = [1,1]^T, \qquad c_3(x,y) = [4x, 1]^T = [-2, 1]^T.$$

which are linear independent, so the LICQ condition is satisfied. The first KKT condition becomes:

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

with solutions $\lambda_2 = 1/3$ and $\lambda_3 = 2/3$, satisfying the last KKT condition. So *yes*, (-0.5,0.5) is a KKT point for the given problem.

b)

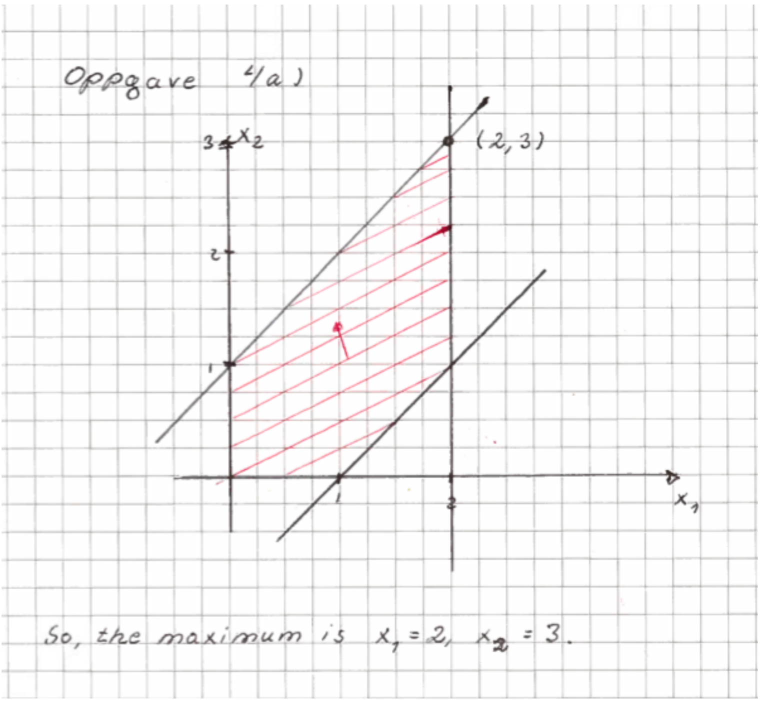


The feasible directions at the KKT point is $d = [\alpha, 1]^T$ with $-1 \leq \alpha \leq 0.5$ (why?). So $\nabla f^T d = -1 + \alpha > 0$ in all feasible directions, and the KKT point is clearly a minimum.

But it is only a local minimum, not a global one.

Problem 4

a)



b)

$$\min \quad -x_1 - 2x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 - x_3 &= -1 \\ -x_1 + x_2 + x_4 &= 1 \\ x_1 - x_5 &= 0 \\ x_1 + x_6 &= 2 \\ x_2 - x_7 &= 0 \\ x_i &\geq 0, \qquad i = 1, 2, \ldots, 7. \end{aligned}$$

Problem 5

a) The Euler-Lagrange equation becomes

$$\frac{d}{dx} x^2 = 2y \qquad \Rightarrow \qquad 2x = 2y \qquad \Rightarrow \qquad y_0(x) = x \in \mathcal{D}$$

b) A functional $J : \mathbb{D} \to \mathbb{R}$ is strictly convex if

$$J(y + v) - J(y) \geq \delta J(y; v), \qquad \forall y, y + v \in \mathcal{D}$$

with equality only if $v = 0$. Here $\delta J(y; v)$ is the Gâteaux variation of J.

In our case, we get

$$\begin{aligned}\delta F(y;v) &= \frac{d}{d\varepsilon}F(y+\varepsilon v)|_{\varepsilon=0} = \int_0^1 (2yv + x^2 v')\,dx \\ &= \int_0^1 2yv\,dx + x^2 v|_0^1 - \int_0^1 2xv\,dx = 2\int_0^1 (y-x)\,dx\end{aligned}$$

since $v(0) = v(1) = 0$, (remember $y + v \in \mathcal{D}$). Further,

$$\begin{aligned}F(y+v) - F(y) &= \int_0^1 (y^2 + 2yv + v^2 + x^2(y' + v') - y^2 - x^2 y')\,dx \\ &= \delta F(y;v) + \int_0^1 v^2\,dx.\end{aligned}$$

The last term is obviously ≥ 0 and equal to zero only if $v = 0$.

Alternatively, see Theorem 3.5 in Troutmann,

Problem 6

a) The extended functional becomes

$$\tilde{F}(y) = \int_0^T \big((\dot{v} + \alpha v)^2 + \lambda v\big)\, dt.$$

The Euler-Lagrange equation is

$$2\frac{d}{dt}(\dot{v} + \alpha v) = 2\alpha(\dot{v} + \alpha v) + \lambda$$

or

$$\ddot{v} - \alpha^2 v = \frac{\lambda}{2}.$$

with boundary conditions

$$v(0) = 0, \qquad \dot{v}(T) + \alpha v(T) = 0.$$

The last one is the open end condition $\tilde{f}_z(T) = 0$. The Lagrange multiplier λ is found from the constraint $G(v) = l$.

b) With $\alpha = 1$ this becomes:

$$\ddot{v} - v = \frac{\lambda}{2} \qquad \Rightarrow \qquad v(t) = C_1 e^{-t} + C_2 e^{t} - \frac{\lambda}{2}.$$

From the boundary conditions we get

$$C_1 = \frac{\lambda}{2}\left(1 - \frac{1}{2}e^{-T}\right), \qquad C_2 = \frac{\lambda}{4}e^{-T}.$$

Finally,

$$G(v) = \int_0^T v(t)\, dt = \lambda\left(\frac{3}{4} - e^{-T} + \frac{1}{4}e^{-2T} - \frac{1}{2}T\right) = l$$

Putting all this toghether, we end up with someting like

$$v_0(t) = \frac{l\left(-2\, e^{-t+2\,T} + e^{-t+T} - e^{T+t} + 2\, e^{2\,T}\right)}{-3\, e^{2\,T} + 4\, e^{T} - 1 + 2\,Te^{2\,T}}$$

The solution is unique since f clearly is strictly convex and g is linear, see Theorem 3.16 in Troutman.