NTNU

Department of Mathematical Sciences

Examination paper for **TMA4180 Optimization Theory**

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Examination time (from–to): 09.00–13.00
Permitted examination support material:

- Approved simple calculator.
- Rottmann: *Matematisk formelsamling*.
- Nocedal & Wright: *Numerical Optimization* + errata.
- Printed lecture notes for the course.

Language: English
Number of pages: 3
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Checked by:

Date Signature

Problem 1 Consider a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula $f(x, y) = 2(x - 3y)^2 + y^4$.

a) Find the point of global minimum of f over $\mathbb{R}^2$.

b) Determine whether the function f is convex or not.

c) Starting from the point $(x, y) = (3, 1)$ take one step of the steepest descent algorithm with linesearch. Use backtracking (Armijo) linesearch (Algorithm 3.1 in Nocedal and Wright). Take the initial step length $\bar{\alpha} = 1$, sufficient decrease parameter $c = 0.25$, and contraction factor $\rho = 0.1$.

Problem 2 De Finetti in 1949 considered the following class of functions:

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *quasi-convex*, if for every $x_1, x_2 \in \mathbb{R}^n$ and every $0 \le \lambda \le 1$ it holds that

$$f(\lambda x_1 + (1 - \lambda)x_2) \le \max\{ f(x_1), f(x_2) \}.$$

a) Show that every convex function on $\mathbb{R}^n$ (not necessarily differentiable) is also quasi-convex.

b) Show that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex *if and only if* for every $\alpha \in \mathbb{R}$ the lower-level set $S_\alpha = \{ x \in \mathbb{R}^n \mid f(x) \le \alpha \}$ is convex.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} 0 & 0 \le x \le 1, \\ 1 & \text{otherwise.} \end{cases}$$

c) Show that f is quasi-convex, but not convex. Additionally, show that a point of local minimum of a quasi-convex function is not necessarily a point of global minimum.

Problem 3 Consider the following variation of the linear conjugate gradient (CG) algorithm for minimizing convex quadratic functions $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi(x) = x^T Ax/2 - b^T x$, $A \in \mathbb{R}^{n\times n}$, $A^T = A$, $b \in \mathbb{R}^n$:

1: Given: initial point $x_0 \in \mathbb{R}^n$
2: Initialization: put $k = 0$, $p_0 = -\nabla\phi(x_0)$.
3: **while** $\nabla\phi(x_k) \not\approx 0$ **do**
4:     Linesearch: $\alpha_k :=$ exact linesearch for ϕ along p_k
5:     Update solution approximation: $x_{k+1} = x_k + \alpha_k p_k$
6:     Update search direction: $p_{k+1} = -\nabla\phi(x_{k+1}) + \beta_{k+1} p_k$
7:     Proceed to next iteration: $k = k + 1$
8: **end while**

In order to update the search direction (step 6), we need to calculate β_{k+1}. *Unlike* in the usual linear CG, in our algorithm we calculate β_{k+1} from the relation $p_{k+1}^T p_k = 0$. Show that the resulting modified CG algorithm is in fact exactly the same as the steepest descent algorithm with exact linesearch.

Problem 4 Consider a set $\Omega \subset \mathbb{R}^2$ defined by two inequality constraints:

$$\Omega = \{\, (x, y) \in \mathbb{R}^2 \mid 25 - x^2 - y^2 \geq 0,\ 4x - 3y \geq 0 \,\}.$$

a) Using a suitable set of linear inequalities and equalities describe the following cones for Ω at $(x, y) = (3, 4)$: (i) cone of linearized feasible directions; (ii) the tangent cone.

b) Determine all values of the parameter π, for which the point $(x, y) = (3, 4)$ is an optimal solution for the following constrained optimization problem:

$$\begin{aligned} \underset{(x,y)}{\text{minimize}} \quad & x + \pi y, \\ \text{subject to} \quad & (x, y) \in \Omega. \end{aligned}$$

Problem 5 Consider the following constrained optimization problem in two real variables x, y:

$$\begin{aligned} \underset{(x,y)}{\text{minimize}} \quad & f(x,y) = \frac{1}{2}(x^2 + y^2), \\ \text{subject to} \quad & x - y - 1 = 0. \end{aligned} \tag{1}$$

a) Find the globally optimal solution (x^*, y^*) for (1) (graphically, if you like). Also find the value of the Lagrange multiplier λ^* associated with the constraint at the globally optimal solution.

b) Formulate the unconstrained minimization problem corresponding to the application of the quadratic penalty method applied to (1). Solve the resulting unconstrained minimization problem for the penalty parameter $\mu = 2$.

Note: $(x - y - 1)^2 = x^2 + y^2 + 1 - 2x + 2y - 2xy$.

c) State the augmented Lagrangian penalty function corresponding to (1) and some unspecified Lagrange multiplier λ and penalty parameter $\mu > 0$. Find the unconstrained global minimum of the augmented Lagrangian corresponding to $\lambda = 0.5$, $\mu = 2$.

Compare the accuracy of the obtained approximate solutions to (1) with those obtained in the previous step.