Exercise #1

January 17, 2023

Problem 1.

a) Prove that the real-valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is lower semi-continuous at $\overline{x} \in \mathbb{R}^n$ if and only if for any $\lambda < f(\overline{x})$ there exists $\delta > 0$ such that $\lambda < f(x)$ for all $x \in \mathbb{B}(\overline{x}, \delta)$, where $\mathbb{B}(\overline{x}, \delta)$ is the open ball with center at $\overline{x}$ and radius δ.

b) Prove that the real-valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is lower semi-continuous at $\overline{x}$ if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that $f(\overline{x}) - \epsilon < f(x)$ for all $x \in \mathbb{B}(\overline{x}, \delta)$. (**Hint:** use the condition given in a).)

c) Prove that the real-valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is lower semi-continuous in $\mathbb{R}^n$ if and only if for any $\lambda \in \mathbb{R}$ the set $L_\lambda = \{x \in \mathbb{R}^n : f(x) > \lambda\}$ is open. (**Hint:** use the condition given in a) or b).)

d) Prove that the real-valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is lower semi-continuous in $\mathbb{R}^n$ if and only if $\operatorname{epi} f$ is closed, where $\operatorname{epi} f$ is the epigraph of the function f, defined as $\operatorname{epi} f = \{(x, p) \in \mathbb{R}^n \times \mathbb{R} : p \geq f(x)\}$. (**Hint:** use the conditions given in a) and c).)

Problem 2.

Check the properties of lower semi-continuity, coercivity and existence of a global minimizer for the following functions:

a) $\ell : \mathbb{R} \mapsto \mathbb{R}$ defined as $\ell(x) = 5x^{10} + 8x^7 - 9x^2 + x + c$, where $c \in \mathbb{R}$ is a constant.

b) $m : \mathbb{R} \mapsto \mathbb{R}$ defined as $m(x) = e^x - \frac{1}{1+x^2}$.

c) $p : \mathbb{R} \mapsto \mathbb{R}$ defined as $p(x) = x^4 - 20x^3 + \sup_{k \in \mathbb{N}} \sin(k^2 x)$.

d) $q : \mathbb{R}^2 \mapsto \mathbb{R}$ defined as $q(x) = x_1^2(1 + x_2^3) + x_1^2$.

Problem 3.

Find the gradient, Hessian, and local minimizers of the objective function f of the optimization problem $\min_{x,y} f(x, y)$, where $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as:

a) $f(x, y) = \frac{x^2}{2} + x \cos y$.

b) $f(x, y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y$.

Problem 4.

Compute the gradient, Hessian and local minimizers of the Rosenbrock function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.

Problem 5.

For a matrix $A \in \mathbb{R}^{d\times d}$, we denote by

$$\|A\|_F := \left(\sum_{i,j=1}^{d} a_{ij}^2\right)^{\frac{1}{2}}$$

its *Frobenius norm*. Show that the optimization problem

$$\min_{\substack{A\in\mathbb{R}^{d\times d},\\ \det A>0}} \left(\|A\|_F + \frac{1}{\det A}\right)$$

admits a global minimum.