# OPTIMISATION WITH CONVEX CONSTRAINTS

PRELIMINARY VERSION

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In this note we will study optimality conditions for constrained optimisation problems of the form

$$(P) \qquad \min_{x\in\Omega} f(x),$$

where $\Omega \subset \mathbb{R}^d$ is some convex set. Unless specified otherwise, we will always assume that:

- The function $f\colon \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^1$.
- The set $\Omega \subset \mathbb{R}^d$ is nonempty, convex, and closed.

## 1. Optimality conditions

In the case of unconstrained optimisation, that is, $\Omega = \mathbb{R}^d$, we have seen that a necessary condition for a point $x^* \in \mathbb{R}^d$ to be a minimiser of $f$ is that $\nabla f(x^*) = 0$. One way to interpret this equation is that all directional derivatives of $f$ at $x^*$ are equal to zero. In other words, if we perturb the point $x^*$ a bit in any direction $p$, the function values do not decrease significantly. Or, we can say that a necessary condition for $x^*$ to be a minimiser of $f$ is that there exists no descent direction $p$ of $f$ at $x^*$.

In the case of constrained optimisation, the situation is notably different, because we do not need to consider every possible direction, but rather only those that—at least for sufficiently small step lengths—do not leave the set $\Omega$ we want to optimise over. These directions are called feasible:

**Definition 1.1.** Let $x \in \Omega$ and $p \in \mathbb{R}^d$. Then $p$ is called a *feasible direction* at $x$, if there exists $t > 0$ such that $x + tp \in \Omega$.

In other words, if we make a sufficiently small step in direction $p$ starting at $x$, we still remain in the set $\Omega$.

At this point, it is important to note that this definition of feasible directions is useful in the context of optimisation only because the set $\Omega$ is assumed to be convex: The convexity of $\Omega$ implies that, given two points contained in $\Omega$, the whole line segment connecting these points is itself completely contained in $\Omega$. Thus, if $p$ is a feasible direction at $x$ and $t > 0$ is such that $x+tp$ is contained in $\Omega$, then $x+\hat{t}p \in \Omega$ for all $0 < \hat{t} < t$ as well.

Moreover, we have the following characterisation of feasible directions:

**Lemma 1.2.** *The direction $p \in \mathbb{R}^d$ is feasible, if and only if $p = t(\hat{x} - x)$ for some $\hat{x} \in \Omega$ and $t > 0$.*

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A large part of this note is based on the note "Introduction to optimality conditions: Optimality conditions for optimization over convex sets" by Anton Evgrafov, NTNU, 2018, which itself was to a large extent based on Section 4.4 in "Introduction to continuous optimization" by N. Andréasson, A. Evgrafov, M. Patriksson, E. Gustavsson, M. Önnheim; Studentlitteratur (2013), 2nd edition.

*Proof.* If $p = t(\hat{x} - x)$ for some $\hat{x} \in \Omega$, then $x + p/t = \hat{x} \in \Omega$, and therefore $p$ is feasible. Conversely, if $p$ is feasible, then $\hat{x} := x + tp \in \Omega$ for some $t > 0$, and we can write $p = (\hat{x} - x)/t$. □

**Proposition 1.3** (First order necessary condition)**.** *Assume that $x^*$ is a local solution of $(P)$. Then*

$$\langle \nabla f(x^*), p \rangle \geq 0 \tag{1}$$

*for all feasible directions $p$ at $x^*$, or, equivalently,*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \tag{2}$$

*for all $x \in \Omega$.*

*Proof.* Assume that $p$ is a feasible direction. Then $x^* + tp \in \Omega$ for all sufficiently small $t > 0$. Since $x^*$ is a local solution of $(P)$, this implies that

$$f(x^*) \leq f(x^* + tp)$$

for all sufficiently small $t > 0$. Thus

$$\langle \nabla f(x^*), p \rangle = \lim_{t \to 0^+} \frac{1}{t}\big(f(x^* + tp) - f(x^*)\big) \geq 0.$$

Due to Lemma 1.2, we can write any feasible direction as $p = t(\hat{x} - x^*)$ for some $t > 0$. Thus the two conditions (1) and (2) are equivalent. □

In the case of convex functions $f$ it turns out that this necessary optimality condition is again sufficient:

**Proposition 1.4** (Necessary and sufficient conditions for convex problems)**.** *Assume that $f$ is convex. Then $x^*$ is a global solution of $(P)$ if and only if $x^* \in \Omega$ and*

$$\langle \nabla f(x^*), p \rangle \geq 0$$

*for all feasible directions $p$ at $x^*$, or, equivalently, that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \tag{3}$$

*for all $x \in \Omega$.*

*Proof.* The necessity and equivalence of these conditions has already been shown in Proposition 1.3. It thus only remains to show that any one of them is sufficient.

To that end, assume that (3) holds and let $x \in \Omega$. The convexity of $f$ implies that

$$f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle.$$

Since by assumption (3) is satisfied, the last term in this inequality is non-negative, and we obtain that $f(x) \geq f(x^*)$. Since this holds for every $x \in \Omega$, it follows that $x^*$ is a global solution of $(P)$. □

Another possibility of formulating optimality conditions is based on the notion of the normal cone, which consists of all direction that form an obtuse angle with all feasible directions at a given point $x$:

**Definition 1.5.** Given $x \in \Omega$, we define the normal cone $N_\Omega(x)$ to $\Omega$ at $x$ by

$$N_\Omega(x) = \big\{q \in \mathbb{R}^d : \langle q, \hat{x} - x \rangle \leq 0 \text{ for all } \hat{x} \in \Omega\big\}.$$

**Proposition 1.6.** *Assume that $x^*$ is a local solution of $(P)$. Then*

$$-\nabla f(x^*) \in N_\Omega(x^*).$$

*Conversely, if $f$ is convex and $-\nabla f(x) \in N_\Omega(x)$, then $x$ is a global solution of $(P)$.*

*Proof.* This is an immediate consequence of the definition of the normal cone $N_\Omega(x)$ and Propositions 1.3 and 1.4. □

Finally, it is possible to define the so called tangent cone to $\Omega$ at the point $x \in \Omega$ by

$$T_\Omega(x) := \{p \in \mathbb{R}^d : \langle q, p\rangle \le 0 \text{ for all } q \in N_\Omega(x)\}. \tag{4}$$

One can show that this tangent cone is (in the convex case!) the same tangent cone as defined in [2, Def. 12.2]. Moreover, the latter can easily be seen to consist (again, only in the convex case!) of all limits of feasible directions at $x$. The proof of these results relies on the notion of polar cones and some results from convex analysis and is quite a bit outside the scope of this note.

**Theorem 1.7.** *The tangent cone $T_\Omega(x)$ to the convex set $\Omega$ at the point $x \in \Omega$ is the closure of the set of all feasible directions at $x$.*

*Proof.* See [1, Chap. III, Cor. 5.2.5]. □

In particular, this implies that the necessary optimality conditions as well as the sufficient optimality conditions for convex functions can be formulated in terms of the tangent cone instead of the set of feasible directions. That is, if $x^*$ is a local solution of $(P)$, then (1) actually holds for all $p \in T_\Omega(x^*)$.

**Remark 1.8.** In this note, we have defined the tangent cone in a somehow roundabout way. More commonly, one starts with defining the tangent cone as the closure of the cone of all feasible directions. Then one introduces the normal cone as

$$N_\Omega(x) = \{q \in \mathbb{R}^d : \langle q, p\rangle \le 0 \text{ for all } p \in T_\Omega(x)\}.$$

Finally, one uses results from convex analysis in order to show that the tangent and normal cone defined in that manner also satisfy (4).

## 2. Projections

Now we consider the special case where $f(x) = \frac{1}{2}\|x - z\|^2$, for some fixed $z \in \mathbb{R}^d$, that is, the problem

$$\min_{x \in \Omega} \frac{1}{2}\|x - z\|^2. \tag{5}$$

In other words, given $z \in \mathbb{R}^d$, we want to find the point $x^* \in \Omega$ for which the (squared Euclidean) distance to $z$ is minimal.

**Lemma 2.1.** *The problem* (5) *has a unique solution.*

*Proof.* The existence of a solution follows from the fact that the function $f(x) = \frac{1}{2}\|x-z\|^2$ is continuous and coercive, and the assumption that $\Omega \subset \mathbb{R}^d$ is non-empty and closed. The uniqueness of the solution follows from the strict convexity of $f$ together with the convexity of $\Omega$. □

**Definition 2.2.** Given $z \in \mathbb{R}^d$, we call the unique solution of (5) the *projection of $z$ onto $\Omega$* and denote it as $\pi_\Omega(z)$.

**Proposition 2.3.** *The projection $\pi_\Omega(z)$ of $z$ onto $\Omega$ is uniquely characterised by the conditions*

$$\pi_\Omega(z) \in \Omega$$

*and*

$$\langle \pi_\Omega(z) - z, x - \pi_\Omega(z) \rangle \ge 0$$

*for every $x \in \Omega$.*

*Proof.* Denote $f(x) := \frac{1}{2}\|x - z\|^2$. Since $f$ is convex, a necessary and sufficient condition for $x^*$ to be a global solution of $\min_{x\in\Omega} f(x)$ is that $x^* \in \Omega$ and

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0$$

for all $x \in \Omega$ (see Proposition 1.4). Now $\nabla f(x^*) = x^* - z$, and thus the necessary and sufficient optimality condition reads as

$$\langle x^* - z, x - x^* \rangle \geq 0 \tag{6}$$

for every $x \in \Omega$. That is, $x^* = \pi_\Omega(z)$, if and only if $x^* \in \Omega$ and (6) holds, which was to show. □

We now return to the problem $(P)$ of minimising an arbitrary function $f$ over a closed and convex set $\Omega$ and use the notion of a projection onto $\Omega$ to formulate yet another characterisation of the solutions.

**Proposition 2.4.** *Assume that $x^*$ is a solution of $(P)$. Then*

$$x^* = \pi_\Omega\big(x^* - \alpha\nabla f(x^*)\big) \tag{7}$$

*for any $\alpha > 0$.*

*Proof.* Since $x^*$ solves $(P)$, it follows that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0$$

for every $x \in \Omega$. As a consequence,

$$\big\langle x^* - (x^* - \alpha\nabla f(x^*)), x - x^* \big\rangle \geq 0$$

for every $x \in \Omega$. As shown in Proposition 2.3, however, this implies that $x^*$ is the projection of $x^* - \alpha\nabla f(x^*)$ onto $\Omega$, or $x^* = \pi_\Omega(x^* - \alpha\nabla f(x^*))$. □

**Remark 2.5.** If $f$ is convex, then it turns out that (7) is both a necessary and sufficient condition for a solution of $(P)$. This readily follows from the fact that, in this case, the variational inequality (3) is a necessary and sufficient optimality condition according to Proposition 1.4.

**Remark 2.6.** Equation (7) implies that every local solution of $(P)$ is a fixed point of the mapping $G\colon \mathbb{R}^d \to \mathbb{R}^d$,

$$G(x) = \pi_\Omega\big(x - \alpha\nabla f(x)\big).$$

As a consequence, it seems reasonable to try to solve $(P)$ by means of the fixed point iteration (the *gradient projection method*)

$$\begin{aligned} z_{k+1} &\leftarrow x_k - \alpha\nabla f(x_k), \\ x_{k+1} &\leftarrow \pi_\Omega(z_{k+1}). \end{aligned}$$

Of course, this only makes sense if the projection on set $\Omega$ can be computed efficiently. In this case, however, this can be a viable, though possibly slow, algorithm for the solution of $(P)$. Indeed, one can show that this algorithm converges if $f$ is a strongly convex $C^1$-function, and the step length $\alpha > 0$ is chosen sufficiently small.

## 3. Optimisation with linear constraints

We now consider specifically the problem of minimising a function $f$ with linear constraints. That is, we assume that we are given a matrix $A \in \mathbb{R}^{m\times d}$ and a vector $b \in \mathbb{R}^m$, and want to solve the problem

$$\min_x f(x) \qquad \text{s.t. } Ax = b. \tag{L}$$

That is, the feasible set $\Omega$ is the affine set

$$\Omega = \big\{x \in \mathbb{R}^d : Ax = b\big\}.$$

Now assume that $x \in \Omega$, that is, that $x$ solves $Ax = b$. Then it is easy to see that a direction $p \in \mathbb{R}^d$ is feasible, if and only if $A(x+p) = b$, that is, that

$$p \in \ker A.$$

Note in particular, that the set of feasible vectors is symmetric in the sense that $p$ is feasible if and only if $-p$ is feasible. As a consequence, we obtain immediately the optimality condition: If $x^*$ solves $(L)$, then $Ax^* = b$ and

$$\langle \nabla f(x^*), p \rangle = 0 \qquad \text{for all} \qquad p \in \ker A.$$

In other words, we have the condition

$$\nabla f(x^*) \in (\ker A)^\perp.$$

However, this can be rewritten as a more useful condition: To that end, recall that

$$(\ker A)^\perp = \operatorname{Ran} A^T.$$

In other words, a vector $p$ is contained in $(\ker A)^\perp$, if and only if it can be written as $p = A^T\lambda$ for some $\lambda \in \mathbb{R}^m$. Thus we obtain the following optimality conditions for $(L)$:

**Lemma 3.1.** *Assume that $x^*$ is a local solution of $(L)$. Then there exists a* Lagrange multiplier *$\lambda^* \in \mathbb{R}^m$ such that*

$$A^T\lambda^* = \nabla f(x^*).$$

*Conversely, if $f$ is convex and there exists $\lambda^* \in \mathbb{R}^m$ such that*

$$\begin{aligned} A^T\lambda^* &= \nabla f(x^*), \\ Ax^* &= b, \end{aligned}$$

*then $x^*$ is a global solution of $(L)$.*

## 4. Concave inequality constraints

Finally, we will discuss the situation where the convex set $\Omega$ is the solution set of a number of inequalities. That is, we are given functions $c_i\colon \mathbb{R}^d \to \mathbb{R}$, $i \in \mathcal{I}$, for some (finite) index set $\mathcal{I}$ and define

$$\Omega = \bigl\{x \in \mathbb{R}^d : c_i(x) \ge 0,\ i \in \mathcal{I}\bigr\}.$$

**Lemma 4.1.** *Assume that the functions $c_i\colon \mathbb{R}^d \to \mathbb{R}$ are concave. Then the set $\Omega$ is convex.*

*Proof.* Assume that $x$, $y \in \Omega$, and that $0 < \lambda < 1$. Then the concavity of the functions $c_i$ implies that

$$c_i(\lambda x + (1-\lambda)y) \ge \lambda c_i(x) + (1-\lambda)c_i(y) \tag{8}$$

for all $i \in \mathcal{I}$. Now the assumption that $x$, $y \in \Omega$ implies that $c_i(x)$, $c_i(y) \ge 0$. Moreover, we have that $\lambda$, $1-\lambda \ge 0$. As a consequence, the right hand side in (8) is non-negative, which in turn shows that

$$c_i(\lambda x + (1-\lambda)y) \ge 0$$

for all $i \in \mathcal{I}$. This, however, shows that $\lambda x + (1-\lambda)y \in \Omega$, and thus $\Omega$ is convex. □

In order to obtain reasonable optimality conditions for optimisation problems with concave inequality constraints, we have to impose additional restrictions on the constraints that guarantee that the tangent and normal cones to $\Omega$ can be easily described by means of the gradients of the constraints. In the general context of constrained optimisation, such conditions are called "constraint qualifications."

**Definition 4.2.** We say that *Slater's constraint qualification* is satisfied, if there exists $\hat{x} \in \mathbb{R}^d$ such that
$$c_i(\hat{x}) > 0 \qquad\qquad \text{for all } i \in \mathcal{I}.$$

In the following, we denote, for $x \in \Omega$, by
$$\mathcal{A}(x) := \{ i \in \mathcal{I} : c_i(x) = 0 \}$$
the set of *active constraints.*

**Theorem 4.3.** *Assume that Slater's constraint qualification holds and that $x \in \Omega$. Then*
$$T_\Omega(x) = \{ p \in \mathbb{R}^d : \langle p, \nabla c_i(x) \rangle \geq 0 \text{ for all } i \in \mathcal{A}(x) \}. \tag{9}$$

*Proof.* Assume first that $p \in T_\Omega(x)$. Theorem 1.7 (and the definition of feasible directions) implies that there exist convergent sequences $x_k \subset \Omega$ and $t_k > 0$ such that
$$p = \lim_{k\to\infty} t_k(x_k - x).$$
Moreover, we note that for every $i \in \mathcal{A}(x)$ we have
$$0 \leq c_i(x_k) \leq c_i(x) + \langle \nabla c_i(x), x_k - x \rangle = \langle \nabla c_i(x), x_k - x \rangle.$$
Thus
$$\langle p, \nabla c_i(x) \rangle = \lim_{k\to\infty} t_k \langle x_k - x, \nabla c_i(x) \rangle \geq 0$$
for all $i \in \mathcal{A}(x)$. That is, every vector $p \in T_\Omega(x)$ has the form given in (9).

Now assume that $p \in \mathbb{R}^d$ is such that $\langle p, \nabla c_i(x) \rangle > 0$ for all $i \in \mathcal{A}(x)$. Then there exists $t > 0$ such that for all $i \in \mathcal{I}$
$$c_i(x + tp) = c_i(x) + t\langle p, \nabla c_i(x) \rangle + o(t) > 0,$$
which implies that $x + tp \in \Omega$. As a consequence, we can write such a vector $p$ as $p = (\tilde{x} - x)/t$ with $\tilde{x} = x + tp \in \Omega$. This shows that all vectors $p \in \mathbb{R}^d$ with $\langle p, \nabla c_i(x) \rangle > 0$ for all $i \in \mathcal{A}(x)$ are feasible directions at $x$.

Finally, let $p \in \mathbb{R}^d$ be such that $\langle p, \nabla c_i(x) \rangle \geq 0$ for all $i \in \mathcal{A}(x)$. Let moreover $\hat{x} \in \Omega$ be such that $c_i(\hat{x}) > 0$ for all $i \in \mathcal{I}$ (such a point exists because of Slater's constraint qualification) and define
$$p_k := p + \frac{1}{k}(\hat{x} - x).$$
Then
$$\langle \nabla c_i(x), p_k \rangle = \langle \nabla c_i(x), p \rangle + \frac{1}{k}\langle \nabla c_i(x), \hat{x} - x \rangle \geq \frac{1}{k}\langle \nabla c_i(x), \hat{x} - x \rangle. \tag{10}$$
However, because of the concavity of $c_i$ we have that
$$0 < c_i(\hat{x}) \leq c_i(x) + \langle \nabla c_i(x), \hat{x} - x \rangle = \langle \nabla c_i(x), \hat{x} - x \rangle$$
for all $i \in \mathcal{A}(x)$. Together with (10), this shows that
$$\langle \nabla c_i(x), p_k \rangle > 0$$
for all $i \in \mathcal{A}(x)$, which in turn shows that all the vectors $p_k$ are feasible directions at $x$. As a consequence, $p$ is the limit of a sequence of feasible directions at $x$. Using Theorem 1.7, we obtain that $p \in T_\Omega(x)$. □

**Theorem 4.4** (Farkas' Lemma)**.** *Let $s_j$, $j \in \mathcal{J}$, be a finite set of vectors in $\mathbb{R}^d$, and let $g \in \mathbb{R}^d$. Then exactly one of the following statements is true:*

*(1) There exist $\lambda_j \geq 0$, $j \in \mathcal{J}$, such that*
$$\sum_{j\in\mathcal{J}} \lambda_j s_j = g.$$

*(2) There exists* $p \in \mathbb{R}^d$ *such that*

$$\langle g, p\rangle < 0$$

*and*

$$\langle s_j, p\rangle \geq 0$$

*for all* $j \in \mathcal{J}$.

*Proof.* See e.g. [2, Lemma 12.4] or [1, Sec. III.4.3]. □

**Theorem 4.5.** *Assume that Slater's constraint qualification holds and that* $x^*$ *is a local solution of the problem* $(P)$*. Then there exists a* Lagrange multiplier $\lambda^* \in \mathbb{R}^{\mathcal{I}}$ *such that*

$$\begin{aligned} &\nabla f(x^*) = \sum_{i\in\mathcal{I}} \lambda_i^* \nabla c_i(x^*), && \\ &\lambda_i \geq 0, && i \in \mathcal{I}, \\ &\lambda_i = 0, && i \notin \mathcal{A}(x^*). \end{aligned} \tag{11}$$

*Conversely, if additionally* $f$ *is convex and* (11) *holds, then* $x^*$ *is a global solution of* $(P)$*.*

*Proof.* Since $x^*$ is a local solution of the problem $(P)$, it follows that $\langle \nabla f(x^*), p\rangle \geq 0$ for all $p \in T_\Omega(x^*)$. Theorem 4.3 implies that this is equivalent to stating that $\langle \nabla f(x^*), p\rangle \geq 0$ for all $p \in \mathbb{R}^d$ with $\langle \nabla c_i(x^*), p\rangle \geq 0$, $i \in \mathcal{A}(x^*)$. In other words, there does not exist a vector $p \in \mathbb{R}^d$ such that $\langle \nabla f(x^*), p\rangle < 0$ and $\langle \nabla c_i(x^*), p\rangle \geq 0$ for all $i \in \mathcal{A}(x^*)$. Thus Theorem 4.4 implies that we can write

$$\nabla f(x^*) = \sum_{i\in\mathcal{A}(x^*)} \lambda_i^* \nabla c_i(x^*)$$

for some $\lambda_i^* \geq 0$, $i \in \mathcal{A}(x^*)$. Setting $\lambda_i^* = 0$ for $i \in \mathcal{I} \setminus \mathcal{A}(x^*)$, we obtain the desired representation of $\nabla f(x^*)$.

The converse direction follows from the fact that the condition $\langle \nabla f(x^*), p\rangle \geq 0$ for all $p \in T_\Omega(x^*)$ is a sufficient optimality condition in the convex case. □

**Remark 4.6.** One can generalise these results to the case where $\Omega$ is given by concave inequality constraints and linear equality and inequality constraints:

$$\Omega = \bigl\{x \in \mathbb{R}^d : c_i(x) \geq 0, i \in \mathcal{I}, \text{ and } Ax \geq b,\ Cx = d\bigr\},$$

with $c_i \colon \mathbb{R}^d \to \mathbb{R}$ concave, $A \in \mathbb{R}^{m\times d}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{\ell\times d}$, $d \in \mathbb{R}^\ell$. In such a case, Slater's constraint qualification reads as: There exists $\hat{x} \in \mathbb{R}^d$ with $c_i(\hat{x}) > 0$, $i \in \mathcal{I}$, and $A\hat{x} \geq b$, $Cx = d$. Put differently, there exists a point $\hat{x}$ that is feasible and that satisfies all non-linear constraints with a *strict* inequality. If this condition is satisfied, one can show (by essentially following the same argumentation as above) that a necessary optimality condition is the existence of $\lambda^* \in \mathbb{R}^{\mathcal{I}}$, $\mu^* \in \mathbb{R}^m$, and $\nu^* \in \mathbb{R}^\ell$ such that

$$\nabla f(x^*) = A^T\mu^* + B^T\nu^* + \sum_{i\in\mathcal{I}} \lambda_i^* \nabla c_i(x^*)$$

with $\lambda_i^* \geq 0$ for all $i \in \mathcal{I}$ and $\lambda_i^* = 0$ whenever $c_i(x^*) > 0$, and $\mu_i^* \geq 0$ for all $i = 1, \ldots, m$ and $\mu_i^* = 0$ whenever $(Ax^*)_i > 0$.

## References

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