PROJECT DESCRIPTION

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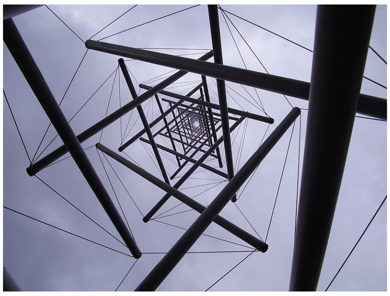
1. Tensegrity Structures

Tensegrity structures are mechanical structures that are constructed of straight elastic members (called *bars*) and elastic *cables* that are connected at *joints* (or *nodes*), and together guarantee the stability of the structure; the name is a *portmanteau* of *tension* and *integrity*, indicating that the integrity of the structure is to a large degree a consequence of the tension in the cables and not the compression forces in the bars.

Originally, they have been introduced as art works, but have since been used more widely in engineering, in particular as "smart structures," which can change their shape (by adjusting the length of the cables) depending on current requirements. Because the cables allow for rather lightweight constructions, the idea of tensegrity has recently also been pursued for the construction of space based objects like satellites and space probes.



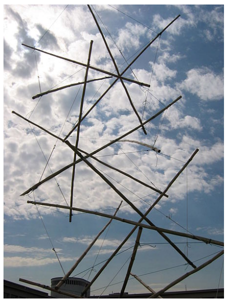


Figure 1. Examples of tensegrity towers. *Left:* Needle Tower II by Kenneth Snelson; Kröller–Müller Museum, Netherlands. *Right:* Photograph of a cane tower built by architecture students at the University of the Witwatersrand.[1]

In this project, you will discuss an optimisation based model for the problem of *form-finding* of a tensegrity structure, that is, the computation of the shape of a structure with a given configuration of bars and cables.

2. Notation and Definitions

In order to describe the configuration of a tensegrity structure, we model it as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, \ldots, N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The vertices represent the joints in the structure, and an edge $e_{ij} = (i, j)$ with $i < j$ indicates that the joints i and j are connected through some member (that is, a bar or a cable).

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[1]Left image by User:Alpha.prim of Wikipedia. Right image by User:Peterleroux of Wikipedia. Both images have been dedicated to the public domain.

The position of node i is denoted as $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \in \mathbb{R}^3$. Moreover, we collect the positions of all nodes in a large vector $X = (x^{(1)}, \ldots, x^{(N)}) \in \mathbb{R}^{3N}$.

The goal of this project is to determine the positions X of all the nodes, that is, the shape of the structure, given its geometry and other physical property. The underlying physical principle for this is that the structure will attain a stable resting position X^* in which the total potential energy has a (local or global) minimum. Thus this position should be computable with methods from optimisation.

In the following, we will discuss the different components of the total energy.

Individual Bars. We assume that all the bars in the structure are made of the same material and have the same thickness and cross section, but may have different lengths.

Consider a bar e_{ij} connecting the nodes i and j. The bar has a resting length $\ell_{ij} > 0$, at which the internal elastic energy is equal to 0. If the bar is either stretched or compressed from this resting length to a new length $L(e_{ij}) = \|x^{(i)} - x^{(j)}\|$, the elastic energy increases. In this project we will use for this energy the quadratic model

$$E_{\text{elast}}^{\text{bar}}(e_{ij}) = \frac{c}{2\ell_{ij}^2}\big(L(e_{ij}) - \ell_{ij}\big)^2 = \frac{c}{2\ell_{ij}^2}\big(\|x^{(i)} - x^{(j)}\| - \ell_{ij}\big)^2,$$

where $c > 0$ is a parameter depending on the material and the cross section of the bar.[2]

In addition, the bar has a gravitational potential energy $E_{\text{grav}}(e)$ that depends on the bar's mass and its z-coordinate. Denoting by ρ the line density of the bar and by g the gravitational acceleration on the earth's surface, we obtain that

$$E_{\text{grav}}^{\text{bar}}(e_{ij}) = \frac{\rho g \ell_{ij}}{2}\big(x_3^{(i)} + x_3^{(j)}\big).$$

Individual Cables. Similarly as for the bars, we assume that all the cables are made of the same material and have the same thickness, but possibly different lengths.

Consider a cable e_{ij} connecting the nodes i and j. The cable has a resting length $\ell_{ij} > 0$ at which the internal elastic energy vanishes. If the cable is stretched, this energy increases. In contrast to the case of the bar, however, "compression" of the cable does not change its internal energy, as the cable simply bends (or becomes slack). Thus the elastic energy is given as

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \dfrac{k}{2\ell_{ij}^2}(\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2 & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \le \ell_{ij}, \end{cases}$$

where $k > 0$ is a material parameter.[3]

In addition, we assume that the weight of the cables is negligible compared to the weight of the bars and can therefore be ignored. Thus we assume that

$$E_{\text{grav}}^{\text{cable}}(e_{ij}) = 0.$$

[2] This is a simple linear elasticity based model of the forces/energies in the bar. If the ratio $L(e_{ij})/\ell_{ij}$ of actual length and resting length of the bar is somewhat close to 1, it is typically a very good approximation to reality. If the compression or tension forces become too large, though, it should be better replaced with a non-linear elasticity model.

[3] Again, we use a linear elasticity model. If the ratio $L(e_{ij})/\ell_{ij}$ of actual length and resting length of the cable is much larger than 1, this might have to be replaced by a non-linear model.

External Loads. In addition to the internal energy of the structure, there also may be an external energy in the form of loads on specific nodes. If we assume that node i is loaded with a mass $m_i \geq 0$, this results in the total external energy of the structure

$$E_{\text{ext}}(X) = \sum_{i=1}^{N} m_i g x_3^{(i)}.$$

Total Energy. Denote now by $\mathcal{B}$, $\mathcal{C} \subset \mathcal{E}$ the bars and cables of the structure, respectively. Then the total energy is given as

$$(1) \qquad E(X) = \sum_{e_{ij} \in \mathcal{B}} \bigl(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})\bigr) + \sum_{e_{ij} \in \mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X).$$

Additional Constraints. The problem of minimising (1) will usually not admit a solution, because the energy E is, in the presence of any gravitational or external energies, unbounded below: We can decrease the total energy of the structure by letting all z-coordinates of all nodes simultaneously tend to $-\infty$. Thus it is necessary to include additional constraints that prevent this from happening. In this project, we will discuss two different types of constraints:

- The first type of constraints is that the positions of some of nodes are fixed, say

$$(2) \qquad x^{(i)} = p^{(i)} \qquad \text{for } i = 1, \ldots, M,$$

  for given $p^{(i)} \in \mathbb{R}^3$ and $1 \leq M < N$. Effectively, this means that the variables $x^{(i)}$ can simply be replaced by the constants $p^{(i)}$ and we obtain a lower dimensional, free optimisation problem.
- The second type of constraints corresponds to a self-supported free standing structure, where the only constraint is that it is above ground. This can be modeled by the inequality constraint

$$(3) \qquad x_3^{(i)} \geq f\bigl(x_1^{(i)}, x_2^{(i)}\bigr) \qquad \text{for all } i = 1, \ldots, N,$$

  where the continuously differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ models the height profile of the ground.

Of course, a combination of these constraints would be possible as well.

Problems.

P1: Show that the problem of minimising (1) with constraints given by (2) admits a solution, provided that the graph $\mathcal{G}$ is connected.

P2: Show that the problem of minimising (1) with constraints given by (3) admits a solution if $f \in C^1(\mathbb{R}^2)$ is coercive. Is it also possible to prove the existence of solutions for the case of a flat ground profile $f(x_1, x_2) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$?

3. Cable-nets

To start with, we consider the simpler situation, where all the members of the structure are cables, that is, the structure is a *cable net*. Moreover, we consider for this case the setting (2), where some of the nodes are fixed. We then end up with the optimisation problem

$$(4) \qquad \min_X E(X) = \sum_{e_{ij} \in \mathcal{E}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \qquad \text{s.t. } x^{(i)} = p^{(i)},\ i = 1, \ldots, M.$$

After replacing the variables $x^{(i)}$, $i = 1, \ldots, M$, in the definition of E by the constants $p^{(i)}$, this becomes a free optimisation problem in the $3(N-M)$ variables $x^{(i)}$, $i = M+1, \ldots, N$.

Problems.

P3: Show that the function E defined in (4) is C^1, but typically not C^2.

P4: Show that the problem (4) is convex. Is the problem strictly convex? Do we necessarily have a unique solution?

P5: Formulate the necessary and sufficient optimality conditions for (4).

P6: Implement a numerical method for the solution of (4).

Test Case. In order to verify your algorithm (and your optimality conditions), you may use the configuration shown in Figure 2 below with four fixed nodes $p^{(1)}, \ldots, p^{(4)}$ and four free nodes $x^{(5)}, \ldots, x^{(8)}$.

Use the following parameters:

- $k = 3$.
- $\ell_{ij} = 3$ for all edges (i, j).
- $p^{(1)} = (5, 5, 0)$, $p^{(2)} = (-5, 5, 0)$, $p^{(3)} = (-5, -5, 0)$, $p^{(4)} = (5, -5, 0)$.
- $m_i g = 1/6$ for $i = 5, 6, 7, 8$.

Then the analytical solution of the problem is given as

$$x^{(5)} = (2, 2, -3/2), \qquad x^{(6)} = (-2, 2, -3/2),$$
$$x^{(7)} = (-2, -2, -3/2), \qquad x^{(8)} = (2, -2, -3/2).$$

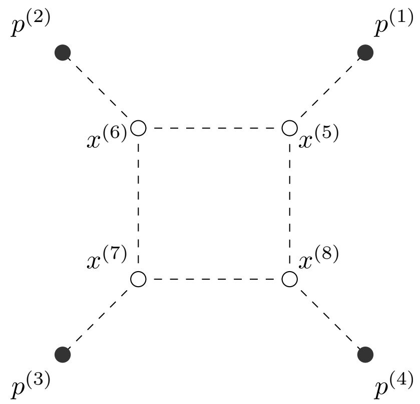

FIGURE 2. Test configuration of a cable net. The black circles indicate fixed nodes $p^{(i)}$, the white circles indicate free nodes $x^{(i)}$. A dashed line between two nodes indicates that these nodes are connected by a cable.

Before you continue with the project, it can be a good idea to test your code (and calculations) thoroughly. Since the optimisation problems you obtain with cable nets are convex, your algorithm should converge to a global solution independent of the initialisation, and the gradient at that solution should be equal to zero up to rounding errors. Also, most optimisation algorithms have fewer potential issues with convex problems; thus, if your algorithm fails or else shows some strange behaviour (like overly slow convergence), this might hint at errors in your implementation.

4. Tensegrity-domes

Next, we consider the situation where we have both bars and cables in the structure, but we still use the constraint of fixed nodes. The resulting structures are in the literature sometimes called *tensegrity-domes*. The resulting optimisation problem reads

$$\begin{aligned}\min_X E(X) = \sum_{e_{ij}\in\mathcal{B}} \big(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})\big) + \sum_{e_{ij}\in\mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X)\\ \text{s.t. } x^{(i)} = p^{(i)},\ i = 1,\ldots,M.\end{aligned} \tag{5}$$

Again, this becomes a free optimisation problem after replacing the variables $x^{(i)}$ with the constants $p^{(i)}$.

Because of the interaction between the tension in the cables and the stresses in the bars, it is now possible to obtain stable structures that rise above the ground. An example of a possible configuration that allows for this to happen is given in Figure 3 below.

Problems.

P7: Show that the function E in (5) is typically not differentiable. Discuss why the lack of differentiability should in practical situations pose no problem.

P8: Formulate the necessary optimality conditions for (5). Are they also sufficient for a local minimum?

P9: Show that the problem (5) is non-convex if $\mathcal{B} \neq \emptyset$. In addition, show by means of an example that (5) may admit (non-global) local minimisers.

P10: Implement a numerical method for the solution of (5).

Test Case. In order to verify your algorithm (and your optimality conditions), you may use the configuration in Figure 3 below with four fixed nodes $p^{(1)}, \ldots, p^{(4)}$ and four free nodes $x^{(5)}, \ldots, x^{(8)}$.

You can use the following parameters:

- $\ell_{15} = \ell_{26} = \ell_{37} = \ell_{48} = 10$.
- $\ell_{18} = \ell_{25} = \ell_{36} = \ell_{47} = 8$.
- $\ell_{56} = \ell_{67} = \ell_{78} = \ell_{58} = 1$.
- $c = 1$.
- $k = 0.1$.
- $g\rho = 0$.
- $p^{(1)} = (1, 1, 0)$, $p^{(2)} = (-1, 1, 0)$, $p^{(3)} = (-1, -1, 0)$, $p^{(4)} = (1, -1, 0)$.
- $m_i g = 0$ for $i = 5, 6, 7, 8$.

Then an analytical solution of the problem is given as[4]

$$x^{(5)} = (-s, 0, t), \qquad x^{(6)} = (0, -s, t),$$
$$x^{(7)} = (s, 0, t), \qquad x^{(8)} = (0, s, t),$$

with

$$s \approx 0.70970 \qquad \text{and} \qquad t \approx 9.54287.$$

You should test your algorithm on more complicated situations as well, though (e.g. non-zero mass of the bars,[5] additional external loads on the free nodes, more complicated configurations; also test your algorithm in non-symmetric situations). The most interesting (local) solutions are those where the free nodes $x^{(i)}$ lie *higher*

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[4]Because of the non-convexity of the problem, there might be (are) other solutions as well, though. Thus your algorithm need not always converge to these same points..

[5]The mass should be small, though. In the configuration above, a value of the size $g\rho = 0.01$ gives a nice result. Much larger values than that yield only "boring" results. Generally, you have to expect that interesting results require a good initialisation.

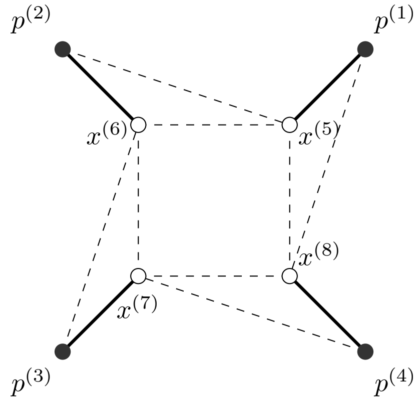

FIGURE 3. Test configuration of a tensegrity dome. The black circles indicate fixed nodes $p^{(i)}$, the white circles indicate free nodes $x^{(i)}$. A dashed line between two nodes indicates that these nodes are connected by a cable. A solid line between two nodes indicates that these nodes are connected by a bar.

than the fixed nodes $p^{(i)}$. Try to come up with initialisations that favor these solutions.

5. Free-standing Structures

Finally, we turn to the case of a free-standing tensegrity structure, where all the nodes are free and the only constraint is that the structure remains above ground. That is, we have the optimisation problem

$$\begin{aligned} \min_X E(X) &= \sum_{e_{ij}\in\mathcal{B}} \bigl(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})\bigr) + \sum_{e_{ij}\in\mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \\ &\text{s.t. } x_3^{(i)} \ge f\bigl(x_1^{(i)}, x_2^{(i)}\bigr), \qquad i = 1,\dots,N. \end{aligned} \tag{6}$$

For this problem to make sense physically, the mass of the bars should be non-zero.

Problems.

P11: Formulate the first order optimality conditions (i.e., KKT-conditions) for (6). Are they sufficient or necessary for a local solution? What can be said about constraint qualifications?

P12: Implement a numerical method for the solution of (6).

Test Case. As a test case, you may use the configuration given in Figure 4 below. This is a modification of the configuration in Figure 3, where the fixed nodes are replaced by free nodes that are held together by additional cables. If you use the parameters from the previous test case, set the cable lengths ℓ_{12}, ℓ_{23}, ℓ_{34}, and ℓ_{14} to 2 set the parameter $g\rho$ to a small positive value (e.g. $g\rho = 10^{-4}$), and use a height profile f that is not too steep (e.g. $f(x_1, x_2) = (x_1^2 + x_2^2)/20$), you can obtain obtain a solution that is very close to the optimum in the previous case.[6]

Feel free to come up with larger examples on your own and test your algorithm on those. One starting point could be to stack two copies of the structure from Figure 4 on top of each other by attaching the lower nodes of the second structure with cables to the upper nodes of the first structure.

[6]Note, though, that this configuration is rotationally symmetric. Depending on your initialisation, your numerical result might therefore at first glance look different.

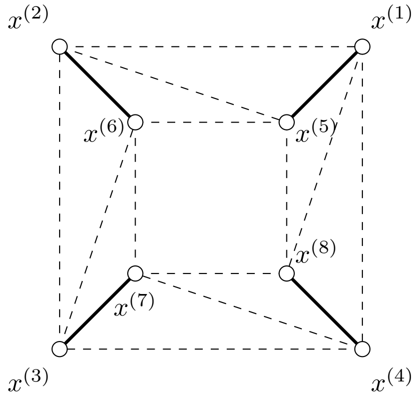

FIGURE 4. Test configuration of a free standing structure. The white circles indicate free nodes $x^{(i)}$. A dashed line between two nodes indicates that these nodes are connected by a cable. A solid line between two nodes indicates that these nodes are connected by a bar.

6. Further Literature and Background

This project was partly inspired by the books [ZO15] and [SdO09], which include a lot of the theoretical background on tensegrity structures. Note, though, that I have been very liberal with introducing new notation that suits the goals of the project. The energy minimisation approach, which the project uses, is described in [ZO15, Chap. 4], mainly in order to discuss the stability properties of the structures (that is, second order optimality conditions). In practice, form finding for tensegrity structures appears to be mostly done by means of the "force-density method", which directly works on the first order optimality condition.

References

[SdO09] Robert E. Skelton and Mauricio C. de Oliveira. *Tensegrity Systems*. Springer, New York, 2009.

[ZO15] Jing Yao Zhang and Makoto Ohsaki. *Tensegrity structures*, volume 6 of *Mathematics for Industry (Tokyo)*. Springer, Tokyo, 2015. Form, stability, and symmetry.

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