TMA4183 - Review on weak formulations of PDEs and finite element methods

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CHAPTER

ONE

RELEVANT CONCEPTS FROM FUNCTIONAL ANALYSIS

Vector space A definition can be found on wiki.

Metric space A metric space is a set X which is equipped with a distance function (or metric)

$$d(x, y) : X \times X \to \mathbb{R}.$$

Complete metric space A metric space is called *complete* if every Cauchy sequence converges to some $x \in X$.

Normed vector space A vector space $(V, \|\cdot\|_V)$ consist of a vector space V which is equipped with a norm

$$\|\cdot\|_V : V \to \mathbb{R}$$

Note that every norm induces a natural metric $d(x, y) := \|x - y\|_V$. Typically we do not use the verbose notation $(V, \|\cdot\|_V)$, instead we simply speak of a normed vector space V, and we omit the subscript $_V$ in the norm symbol when the norm is clear from the context.

Banach space A normed vector space which is complete with respect to the induced metric.

Inner product space An inner product space $\big(V, (\cdot,\cdot)\big)_V$ is a real (or complex) vector space V equipped with a inner product

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R} \quad \text{(or } \mathbb{C}\text{)}$$

Every inner product induces a natural norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$, and thereby a metric. Again, we typically do not use the verbose notation $\big(V, (\cdot,\cdot)\big)$, instead we simply speak of a inner product space V, and we often omit the subscript $_V$ in $(\cdot,\cdot)_V$ symbol when the inner product is clear from the context.

Inner products satisfy the *Cauchy-Schwarz inequality:*

$$(u, v)_V \leqslant \|u\|_V \|v\|_V.$$

Bounded linear operator A linear operator $L : V \to W$ between two normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ is call bounded if there is a constant $C \in \mathbb{R}_0^+$ such that

$$\|Lv\|_W \leqslant C\|v\|_V.$$

The *operator norm* $\|L\|_{V\to W}$ of T is then the smallest such constant given by

$$\begin{aligned}
\|L\| &= \inf\{C \in \mathbb{R}_0^+ : \|Lv\|_W \leqslant \|v\|_V \; \forall v \in V\} \\
&= \sup_{v \in V \; \{0\}} \frac{\|Lv\|_W}{\|v\|_V} \\
&= \sup_{v \in V, \|v\|_V = 1} \|Lv\|_W.
\end{aligned}$$

It can be shown that the the following statements are equivalent for **linear operators**:

- $L : V \to W$ is bounded

- $L : V \to W$ is continuous

Exercise 1

Before you look up the proof, try to prove the previous claim yourself.

A linear operator $l : V \to \mathbb{R}$ (or $\mathbb{C}$) is often called a *linear functional* or a *linear form* on V.

Dual space The dual space V^* for a normed vector space $(V, \|\cdot\|)$ consist of all **continuous** linear functionals defined on V.

Note that for inner product spaces V, every $u \in V$ give rise to a continuous linear functional l_u defined by

$$l_u(v) := (v, u)_V \quad \forall v \in V.$$

For Hilbert space H, that is in essence all the continuous linear functionals you can construct on H thanks to the following theorem.

Riesz representation theorem

Theorem 1 (Riesz representation theorem)

Let H be a Hilbert space with a inner product $(\cdot, \cdot)$. Then for every continuous functional $l : H \to \mathbb{R}$, there is a unique vector $u_l \in H$ such that

$$l(v) = (v, u_l) \quad \forall v \in H,$$

and we have that

$$\|l_u\|_{V^*} = \|u\|_V.$$

Proof. For a proof, we refer to Section 5.2 in [Brezis, 2011].

Later, when we have introduced the concept of weak formulation of partial differential equations, we will make heavily use of the Lax-Milgram theorem.

Theorem 2 (Lax-Milgram)

Given a Hilbert space $(V, \|\cdot\|)$, a bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ (or $\mathbb{C}$), and a linear form $l(\cdot) : V \to \mathbb{R}$ (or $\mathbb{C}$). Then the problem: Find $u \in V$ such that

$$a(u, v) = l(v) \quad \forall v \in V \tag{1.1}$$

possesses solution a solution $u \in V$ if the following assumptions are satisfied.

1. The linear form l is bounded, i.e. there exists a constant $C_l \geqslant 0$ such that

$$|l(v)| \leqslant C_l \|v\| \quad \forall v \in V. \tag{1.2}$$

2. The bilinear form a is bounded, i.e. there exists a constant $C_a \geqslant 0$ such that

$$|a(v, w)| \leqslant C_a \|v\| \|w\| \quad \forall v, w \in V. \tag{1.3}$$

3. The bilinear form a is coercive, i.e. there is a constant $\alpha > 0$ such that

$$a(v, v) \geqslant \alpha \|v\|^2 \quad \forall v \in V. \tag{1.4}$$

Moreover, the solution u satisfies the stability (or a priori) estimate

$$\|u\| \leqslant \frac{C_l}{\alpha} \tag{1.5}$$

and is (therefore!) uniquely defined.

Proof. For a complete proof in particular the existence of a solution u, we refer to the nice presentation in [Evans, 2010]. Here, we only show to derive (1.5) and uniqueness of u.

Assume u solves (1.1). Then set $v = u$ and successively employ the coercivity of a and boundedness of l to see that

$$\alpha \|u\|^2 \leqslant a(u, u) = l(u) \leqslant C_l \|u\|.$$

Dividing the previous chain of inequalities by α and $\|u\|$ if $\|u\| \neq 0$ yields (1.5). For $\|u\| = 0$ the stability estimate is trivially satisfied.

If u_1 and u_2 both satisfy problem (1.1), then thanks to linearity of a in the first slot, the difference $u_1 - u_2$ satisfies problem~(1.1) but with $f = 0$ instead. In that case $C_l = 0$ and thus $0 \leqslant \|u_1 - u_2\| \leqslant \frac{0}{\alpha} = 0$, and thus $u_1 = u_2$.

Remark 1

The *Lax-Milgram theorem* ensures that problem (1.1) is well-posed, i.e.,

- **Existence** of a solution
- **Uniquessness** of the solution
- **Continuous dependency on the data** (or **Stability**) of the solution. In the particular case of *Lax-Milgram theorem*, stability is guaranteed through (1.5) which implies that "small changes" in a and l will only lead to small changes in the solution u.

CHAPTER

TWO

A BRIEF REVIEW ON FUNCTION SPACES

In this chapter we collect some results on various function space we will use throughout the book. One essential property of many function space we will consider is that they are *complete*, i.e. they are either Banach or Hilbert space, see Section *Relevant concepts from functional analysis*.

Note: This section barely scratches at the surface of the topic, we will only summarize (and not even prove) the most essential results we need later one in this course.

Also, this chapter will be a work in progress during the entire course, as we will add relevant results here whenever we need them elsewhere.

2.1 Measure and integration theory, Lebesque spaces

Lebesque integration theory provides a powerful generalization of the Riemann integral which makes sure that the set of so-called Lebesque-integrable functions turns into a Banach space when endowed with a suitable norm. Nowdays, in most standard text books, Lebesque integration theory is presented as part of the curriculum on *Measure and Integration theory*, see Chapter 9-10 in [Browder, 2012] for a quick introduction. To this end,

Definition 1 (Lebesque spaces)

Let $\Omega \subset \mathbb{R}^n$ be a open domain.

Then the Lebesque spaces $L^p(\Omega)$ are defined by

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and } \|f\|_{L^p(\Omega)} < \infty\}. \tag{2.1}$$

Here, the L^p-norm $\|\cdot\|_{L^p(\Omega)}$ is defined by

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p} & 1 \leqslant p < \infty \\ \operatorname{ess\,sup}_\Omega |u| & p = \infty \end{cases} \tag{2.2}$$

Sometimes we write $\|f\|_{p,\Omega}$ instead of $\|f\|_{L^p(\Omega)}$. A function $f \in L^p(\Omega)$ is often called L^p-integrable.

We also introduce the space of *locally* L^p-integrable functions on Ω; that is, functions that are L^p integrable on every compact subset $K \Subset \Omega$,

$$L^p_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} | f \in L^p(K)\ \forall K \Subset \Omega\}. \tag{2.3}$$

TODO

Introduce inner product on L^2.

Lemma 1 (Determining uniqueness through testing)

Let $u_1, u_2 \in L^1_{\text{loc}}(\Omega)$ and assume that

$$\int_\Omega (u_1 - u_2)\phi \,\mathrm{d}x = 0 \quad \forall \phi \in C_c^\infty(\Omega).$$

Then $u_1 = u_2$ almost every in Ω; that is, up to set of measure 0.

Remark 2

In this setting, ϕ is typically called a *test function*. When determining whether two functions are equal, the previous lemma roughly states that you can do this by comparing their "actions" on suitable test functions ϕ instead of comparing their values at (almost) every point.

Here, the "action" is simply the resulting number computed from multiplying the functions in question with the test function ϕ and integrating over Ω.

2.2 Sobolev spaces

2.2.1 Weak derivatives

Let start with a motivating example. Let $u \in C^k(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. Using Green's theorem and taking into account that $\phi = 0$ on a open neighborhood of the boundary of Ω, we see that

$$\int_\Omega \partial_{x_i} u \phi \,\mathrm{d}x = -\int_\Omega u \partial_{x_i} \phi \,\mathrm{d}x, \tag{2.4}$$

and iterating this formula, we observe that for any multiindex $\alpha \in \mathbb{N}^n$ with $\alpha \leqslant k$, it holds that

$$\int_\Omega \partial^\alpha u \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi \,\mathrm{d}x, \tag{2.5}$$

where $|\alpha| = \alpha_1 + \cdots \alpha_n$. Note that the integral expression on the right-hand side of (2.5) makes perfectly sense even for $u \in L^1_{\text{loc}}$ and not only $u \in C^k(\Omega)$. This leads to a possibility to generalize or weakened the classical definition of derivatives.

Definition 2 (Weak derivative)

Let $\alpha \in \mathbb{N}^n$ be a multiindex and $u, u_\alpha \in L^1_{\text{loc}}(\Omega)$. We say that u_α is *α-th weak derivative* of u if

$$\int_\Omega u_\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi \,\mathrm{d}x$$

holds for all $\phi \in C_c^\infty(\Omega)$.

Lemma 2 (Uniqueness of weak derivatives)

If $u \in L^1_{\text{loc}}(\Omega)$ possesses an α-th weak derivative, it is uniquely defined in $L^1_{\text{loc}}(\Omega)$.

Proof. For two weak derivatives u_α and $\tilde{u}_\alpha$ we have that

$$\int_\Omega u_\alpha \phi = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi dx \tag{2.6}$$

$$\int_\Omega \tilde{u}_\alpha \phi = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi dx \tag{2.7}$$

and by substracting the second from the first inequality, we obtain that

$$\int_\Omega (u_\alpha - \tilde{u}_\alpha)\phi dx = 0 \quad \forall \phi \in C_c^\infty(\Omega),$$

and thus $u_\alpha = \tilde{u}_\alpha$ almost everywhere by *Lemma 1*.

Exercise 2 (Relation between the modulus function and the Heaviside function)

Let $\Omega = (-1, 1)$ and set

$$u(x) = |x|$$

$$H(x) = \begin{cases} -1 & x \in (-1, 0) \\ 1 & x \in [0, 1) \end{cases}$$

By simply using the definition of the weak derivative, show that $H(x)$ is the weak derivative of u.

Definition 3 (Sobolev spaces)

- $W^{k,p}(\Omega) := \{u \in L^p(\Omega) |\, \partial^\alpha u$ exists and belongs to $L^p(\Omega)\ \forall \alpha$ with $|\alpha| \leqslant k\}$
- For $p = 2$, we usually write

  $$H^k(\Omega) := W^{k,2}(\Omega)$$

  Note that the $\|\cdot\|_{H^k(\Omega)}$ is induced by the inner product

  $$(v, w)_{H^k(\Omega)} := \sum_{|\alpha| \leqslant k} (\partial^\alpha v, \partial^\alpha w)_{L^2(\Omega)}$$

- For $u \in W^{k,p}(\Omega)$, we set

  $$\|u\|_{W^{k,p}(\Omega)} := \|u\|_{k,p,\Omega} := \begin{cases} \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p} & 1 \leqslant p < \infty, \\ \sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^\infty(\Omega)} & p = \infty. \end{cases}$$

- We set

  $$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{k,p,\Omega}},$$

  that is, the topological closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

- Finally, we introduce the common notation for the dual space of $H^1_0(\Omega)$,

$$H^{-1}(\Omega) := (H^1_0(\Omega))'.$$

Remark 3

$W^{k,p}_0(\Omega)$ can be understood as the closed subspace consisting of those function ϕ in $W^{k,p}(\Omega)$ which are limits of sequences $\{\phi_n\}_{n=1}^{\infty} \subset C^{\infty}_c(\Omega)$.

Later we will need the following important result known as Poincaré inequality.

Theorem 3 (Poincaré inequality)

Let Ω be an open and bounded subset of $\mathbb{R}^n$ and suppose then there is a constant $C_P = C_P(p, n, \Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leqslant C_P \|\nabla u\|_{L^p(\Omega)}.$$

for any $u \in W^{1,p}_0(\Omega)$.

Proof. For a proof we refer to [Evans, 2010] (p. 279).

Corollary 1

On $W^{1,p}_0(\Omega)$, the $\|\cdot\|_{W^{1,p}(\Omega)}$ is equivalent to the norm

$$\|u\|_{W^{1,p}_0(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$$

Proof. A simple application of the Poincaré application yields

$$\|\nabla u\|^p_\Omega \leqslant \|u\|^p_\Omega + \|\nabla u\|^p_\Omega \leqslant (1 + C^p_P)\|\nabla u\|^p_\Omega.$$

2.2.2 Trace operators

Next, we very briefly discuss whether and how functions of certain Sobolev spaces defined on the domain Ω can be restricted to the boundary $\partial\Omega$. This plays an important role in the well-posedness of boundary value problems, as we need to determine the correct spaces for the boundary data in a e.g. Dirichlet or Neumann boundary problem when the data is **non-homogeneous**.

For the remaining part of this Chapter, we assume that Ω is bounded and has a "well-behaving" boundary, that is, it is either Lipschitz or — if this doesn't tell you much — is simply C^{∞}.

Theorem 4 (Traces of $H^1(\Omega)$ spaces)

For a bounded domain Ω with Lipschitz (or C^{∞}) boundary $\Gamma = \partial\Omega$, there exists a bounded operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ (the so-called *Trace Operator*) such that $\gamma(u) = u|_{\Gamma}$ whenever $u \in C(\overline{\Omega})$.

If such a trace operator exists, then one can show that

$$H_0^1(\Omega) = \ker\gamma = \{v \in H^1(\Omega) \mid \gamma(v) = 0\}.$$

It turns out that the trace operator γ **is not onto** $L^2(\Omega)$. Thus, when we later want to find certain weak formulations and solutions $u \in H^1(\Omega)$ which also need to satisfy certain inhomogeneous boundary conditions such as $u = u_D$ on Γ, we need to be careful about the choice of function space from which we take the boundary data u_D. That motivates the following

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Definition 4 ($H^{1/2}(\Gamma)$)

We set

$$H^{1/2}(\Omega) = \{v \in L^2(\Omega) \mid \gamma(\overline{v}) = v \text{ for some } \overline{v} \in H^1(\Omega)\}$$

and define a corresponding norm by

$$\|v\|_{H^{1/2}(\Gamma)} := \|v\|_{1/2,\Gamma} := \inf\{\|\overline{v}\|_{1,\Omega} \mid \gamma(\overline{v}) = v\}.$$

Consequently,

$$\|v\|_{1/2,\Gamma} \leqslant \|v\|_{1,\Omega}.$$

CHAPTER

THREE

WEAK FORMULATION OF PARTIAL DIFFERENTIAL EQUATIONS

In this chapter, we briefly discuss how the functional analysis and function space apparatus can be employed to analyse the well-posedness of certain class of PDEs when given in a so-called "weak" formulation. We start by considering the Poisson problem

$$\nabla \cdot \nabla u = -\Delta u = f \quad \text{in } \Omega \tag{3.1}$$

supplemented with some suitable boundary conditions which u should satisfy on the boundary $\Gamma = \partial\Omega$ of Ω.

The PDE (3.1) is the prototype example of a 2nd order elliptic operator. More generally and without any significant complications, we can consider a more general PDE of the form

$$\mathcal{A}u := -\nabla \cdot (A\nabla u) = f$$

where $A = (a_{ij}(x))_{i,j=1}^n$ is a pointwise defined matrix. Note that

$$\nabla \cdot (A(x)\nabla u(x)) = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u(x)) \tag{3.2}$$

For most part of the remaining lectures, we will require $A(x)$ to satisfy the following defintion.

Definition 5 (Ellipticity of $\mathcal{A}$)

The partial differential operator $\mathcal{A}$ given by (3.2) with coefficients $A = (a_{ij})_{i,j=1}^N \in (L^\infty(\Omega))^{n\times n}$ is called elliptic constant $\alpha > 0$ such that

- $\lambda \cdot A(x)\lambda \geqslant \alpha|\lambda|^2$

for any $\lambda \in \mathbb{R}^n$.

Remark 4

Note that $A \in (L^\infty(\Omega))^{n\times n}$ also implies that that there exists an $\beta \geqslant 0$ such that also

- $|A(x)\lambda| \leqslant \beta|\lambda|$

holds for any $\lambda \in \mathbb{R}^n$, and by ellipticity, we can conclude that in fact $\beta \geqslant \alpha > 0$.

Exercise 3

Prove the statements made in the previous remark

TODO

- Relate $\mathcal{A}$ to classical Poisson problem
- Explain why general $A(x)$ is useful, e.g. anisotropic heat conduction problems

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We now prepared to investigate the well-posedness of a number of boundary value problems where we supplement the partial differential operator $\mathcal{A}$ with one of the following boundary conditions

- **Dirichlet boundary conditions** Given function $g_D : \Gamma \to \mathbb{R}$, we require that

$$u = u_D \quad \text{on } \Gamma$$

- **Neumann boundary conditions** Given function $g_N : \Gamma \to \mathbb{R}$, we require that

$$\mathbf{n} \cdot A\nabla u = g_N \quad \text{on } \Gamma$$

- **Robin boundary conditions** Given functions $g_R, \sigma : \Gamma \to \mathbb{R}$, we require that

$$\mathbf{n} \cdot A\nabla u = \sigma(g_R - u) \quad \text{on } \Gamma$$

These boundary conditions are called *homogeneous* if g_D (respectively g_N, g_R) is zero, otherwise we deal with *inhomogeneous* boundary data. We start by looking at the Poisson supplemented with Neumann boundary conditions

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TODO

Later, mention possible impact on test and trial spaces.

3.1 Neumann problems

Let us consider the homogenous Neumann problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \Gamma \end{cases} \tag{3.3}$$

Here, we used the slightly simplified notation $\partial_n u = \mathbf{n} \cdot \nabla u$. The idea to derive a so-called weak formulation of an PDE is very similar to the idea behind the introduction of weak derivatives: We multiply with a suitable test function v, integrate over Ω and perform integration by parts to transfer a number of derivatives to the test function v.

What kind of test function space we choose is often dictated by 2 considerations:

1. What kind of smoothness do we require to make the derived formulation work?
2. How do we take into account the boundary conditions?

For the Neumann boundary problem, let's assume for the moment that u, our boundary Γ and our test functions v are smooth enough so that we can use Green's theorem, e.g, $u \in C^2(\overline{\Omega})$, Γ is a C^1 boundary, and $v \in C^\infty(\overline{\Omega})$. Then multiplying the PDE in (3.1) with v and integrating over Ω and applying Green's theorem leads to

$$\begin{aligned} \int_\Omega f v \,\mathrm{d}x &= -\int_\Omega \nabla \cdot (\nabla u) v \,\mathrm{d}x + \int_\Omega u v \,\mathrm{d}x \\ &= -\int_\Gamma \underbrace{(\mathbf{n} \cdot \nabla u)}_{=0} v \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega u v \,\mathrm{d}x \end{aligned} \tag{3.4}$$

Note that the Neumann boundary condition $\mathbf{n} \cdot \nabla u = 0$ makes the boundary integrals vanish. Also observe that the right-hand side of (3.4) can be interpreted as taking the inner product associated with $H^1(\Omega)$ between u and v. In fact, the expression makes perfectly sense even if we assume only assume that both $u, v \in H^1(\Omega) =: V!$. With this assumption, we can define the bilinear form

$$a(v, w) := \int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x + \int_\Omega vw \, \mathrm{d}x \tag{3.5}$$

on $V \times V$, and it is straightforward to show that $a(\cdot, \cdot)$ (being the H^1 inner product itself) satisfies the required assumptions of *the Lax-Milgram theorem*:

$$\text{Boundedness: } a(v, w) := \int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x + \int_\Omega vw \, \mathrm{d}x = (v, w)_{H^1(\Omega)} \leqslant \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \tag{3.6}$$

$$\text{Coercivity: } a(v, v) = \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \int_\Omega |v|^2 \, \mathrm{d}x = (v, v)_{H^1(\Omega)} = \|v\|^2_{H^1(\Omega)} \tag{3.7}$$

Next, we define the linear form $l : V \to \mathbb{R}$

$$l(v) := \int_\Omega f v \, \mathrm{d}x = (f, v)_{L^2(\Omega)} \tag{3.8}$$

If we assume that $f \in L^2(\Omega)$, then thanks to the Cauchy-Schwarz inequality,

$$|l(v)| = |(f, v)_{L^2(\Omega)}| \leqslant f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leqslant f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)},$$

we can immediately conclude that l is a continuous bilinear form with $C_l = \|f\|_{L^2(\Omega)}$. Thus the *the Lax-Milgram theorem* let us conclude that the problem: find $u \in H^1(\Omega) =: V$ such that $\forall v \in V$

$$a(u, v) = l(v)$$

has a unique solution for every $f \in L^2(\Omega)$ with $\|u\|_{H^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$.

TODO (variants of Neumann problems)

Discuss Neumann problems when a) lower term is not present b) $g_N \neq 0$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \Gamma \end{cases} \tag{3.9}$$

and

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_n u = g_N & \text{on } \Gamma \end{cases} \tag{3.10}$$

3.2 Robin problems

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \Gamma \end{cases} \tag{3.11}$$

3.3 Dirichlet problems

3.3.1 Homogeneous Dirichlet problem for $-\Delta + \mathrm{Id}$ operator

Next, we consider

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \tag{3.12}$$

We proceed as for the Neumann problem: we multiply with suitable test functions v and integrate by part, but this time, the boundary integral does not vanish since we don't have natural boundary conditions to incorporate. To compensate, we only consider test functions $v \in C_c^\infty(\Omega)$ which vanish at the boundary. Then again, we obtain

$$\begin{aligned} \int_\Omega f v \,\mathrm{d}x &= -\int_\Omega \nabla \cdot (\nabla u) v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x \\ &= -\int_\Gamma (\mathbf{n} \cdot \nabla u) \underbrace{v}_{=0} \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Intuitively speaking we know how the solution u is going to look like on the boundary, namely $u = 0$, so we don't need test functions which test for how the equation "behaves" at the boundary. Also, we now require that our function u comes from a function space where the boundary condition $u = 0$ is already incorporated. This is exactly what the $H_0^1(\Omega)$ space is made for! So the weak formulation for (3.3.1) is:

Find $u \in V := H_0^1(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in V, \tag{3.14}$$

where $a(\cdot, \cdot)$ and $l(\cdot)$ are defined as in (3.5) and (3.8), respectively. As in the case for the homogeneous Neumann problem (3.1), we can show that a and l satisfy the assumption of the *the Lax-Milgram theorem*, and therefore we can conclude there there is a unique solution u to the weak formulation of the homogeneous Poisson problem which depends continuously on the data f.

Important: The only but very important difference between the weak formulation of the *homogeneous Neumann* problem (3.1) and the *homogeneous Dirichlet* problem (3.3.1) is the Hilbert space on which they are posed on.

3.3.2 Homogeneous Dirichlet problem for $-\Delta$ operator

Now, we consider a slightly modified problem Poisson problem where the low order term u is left out:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \tag{3.15}$$

Repeating the steps from the previous section, we arrive at the problem: Find $u \in V := H_0^1(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in V, \tag{3.16}$$

with the only distinction that $a(\cdot, \cdot)$ is now given by

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w dx.$$

The boundedness of $a(\cdot, \cdot)$ and $l(\cdot)$ can be shown (almost) exactly as before. But let's have a look at the coercivity/ellipticity: Setting $u = v$, we obtain

$$a(v, v) = \int_\Omega |\nabla v|^2$$

But thanks to the *Poincaré inequality* and *Corollary 1* we not only know that $|\nabla\ \cdot|$ defines norm on the closed subspace $H_0^1(\Omega)$ but that this norm is equivalent to the usual H^1-norm. Thanks to the proof of *Corollary 1* we see that

$$a(v,v) = \int_\Omega |\nabla v|^2 \geqslant (1+C_p^2)^{-1/2}\|u\|_{1,\Omega}^2.$$

3.3.3 Inhomogeneous Dirichlet problem for $-\Delta$ operator

Next, we consider

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = g_D & \text{on } \Gamma \end{cases} \tag{3.17}$$

Compared to our previous weak formulation for the homogenous, the main question is now: how can we incorporate the non-homogenous Dirichlet b.c. $u = g_D$? First, we realize that the trial function $H_0^1(\Omega)$ for the solution does not make sense anymore. So let's start from $H^1(\Omega)$. Then we also observe that the data g_D should be in $H^{1/2}(\Gamma)$, see *Definition 4* to ensure that we can satisfy the equation $u = g_D$, and only u satisfying this b.c. should be vialable solution candidates for our weak formulation. Thus we set

$$H_{g_D}^1(\Omega) := \{v \in H^1(\Omega) \mid \gamma(v) = g_D\}.$$

Since $g_D \in H^{1/2}(\Gamma)$, this set is not empty. Note that $H_{g_D}^1(\Omega)$ is not really a vector space whenever g_D is not 0 everywhere since the addition of two functions u_1, u_2 with the same non-vanishing boundary data g_D will result in a function u satisfying $u = 2g_D$! Is that sense, $H_{g_D}^1(\Omega)$ should rather be considered as **affine** subspace: For any u_{g_D} satfying $\gamma(u_{g_D}) = g_D$, it holds that

$$H_{g_D}^1(\Omega) = u_{g_D} + H_0^1(\Omega) := \{u_{g_D} + v \mid v \in H_0^1(\Omega)\} = \gamma^{-1}(g_D).$$

So the resulting weak formulation is Find $u \in V := H_{g_D}^1(\Omega)$ such that for all $v \in \widehat{V} := H_0^1(\Omega)$,

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{=:a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{=:l(v)}.$$

Note how in this case the trial function space and test function space are not identical any more! How can we prove the well-posedness of this weak formulation? Lax-Milgram usually requires that the first and second slot of $a(\cdot,\cdot)$ invokes elements from the same (vector) space! The common trick here is to "**lift**" the boundary condition, i.e. we know that by the definition of $H^{1/2}(\Gamma)$ there must a $u_g \in H^1(\Omega)$ such that $\gamma_{u_g} = g_D$. Then we make the ansatz $u = u_0 + u_g$ and with $u_0 \in H_0^1(\Omega)$, leading to the following weak formulation: find $u_0 \in H_0^1(\Omega) =: V$ such that

$$a(u_0, v) = l(v) - a(u_g, v) =: \tilde{l}(v) \forall\, v \in V.$$

CHAPTER

FOUR

A PRIMER ON FINITE ELEMENT METHODS

4.1 Galerkin's method: A recipe to discretize partial differential equations

4.1.1 The general recipe

Galerkin's method is an general approach to solve partial differential equation numerically by transforming them into a system of discrete equations. The computed solution to the discrete equations can then be thought of as an approximation to the solution of the original PDE. Figure Fig. 4.1 summarizes the four stages of the discretization approach which we describe next. As usual, we will use the Poisson problem as a guiding prototype example.

Note: The following recipe is deliberately kept vague. Details such as boundary conditions, suitable function spaces, and

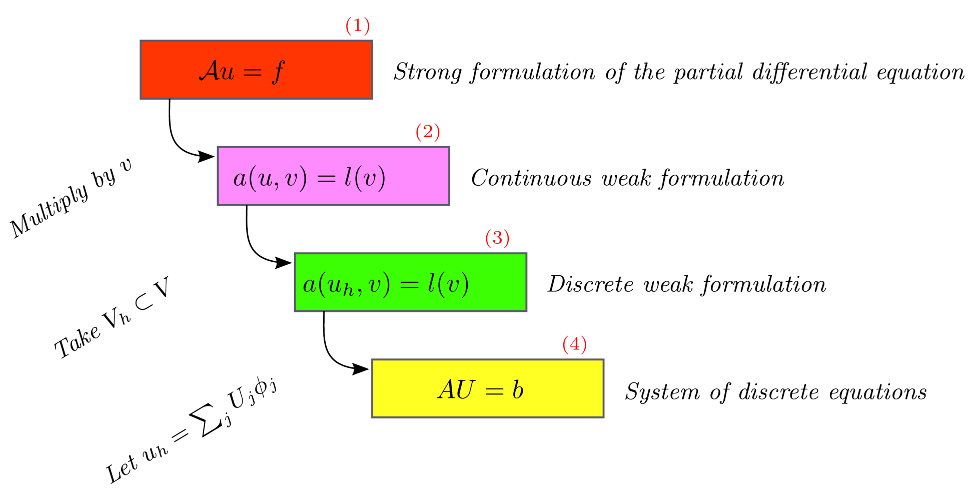

Fig. 4.1: The four stages of Galerkin's method to discretize partial differential equations.

Stage 1 Strong formulation of the PDE

Starting point is a partial differential equation

$$\mathcal{A}u = f \tag{4.1}$$

in *strong form*: for given function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and partial differential operator $\mathcal{A}$, we assume that the function $u : \Omega \to \mathbb{R}$ satisfies the relation (4.1) *pointwise* so that $\mathcal{A}u(x) = f(x)\ \forall x \in \Omega$.

Our favorite example is of course the Poisson problem with *homogeneous Dirichlet boundary condition*:

$$-\Delta u = f \quad \text{in } \Omega, \tag{4.2}$$

$$u = 0 \quad \text{on } \Gamma. \tag{4.3}$$

Stage 2 Continuous weak formulation of the PDE

Find $u \in V$ such that

$$a(u, v) = l(v) \quad \forall v \in V. \tag{4.4}$$

The standard approach to obtain a weak formulation is to multiply with the strong form of the PDE with appropriate test functions which satisfy appropriate smoothness assumptions and —if required— boundary conditions.

For the Poisson problem with homogeneous Dirichlet b.c. we saw previously that we arrived at the weak formulation: Find $u \in V := H^1_0(\Omega)$ s.t. $\forall v \in V$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{l(v)}.$$

Stage 3 Discrete weak formulation of the PDE

By choosing a suitable approximation space $V_h \subset V$ with finite dimension $N = \dim V_h$, we obtain the discrete weak formulation:

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h. \tag{4.5}$$

Stage 4 Formulation as system of discrete equations

To translate the now finite-dimensional problem (4.5) into a discrete system of equations, we choose a basis for the discrete function space

$$V_h = \text{span}\{\phi_i\}_{i=1}^N.$$

First observe that the problem (4.5) is then equivalent to seek a $u_h \in V_h$ such that

$$a(u_h, \phi_h) = l(\phi_i) \quad i = 1, \dots N, \tag{4.6}$$

since $a(\cdot, \cdot)$ and $l(\cdot)$ are linear in their respectively second and first slot.

The second step is now to rewrite $u_h = \sum_{j=1}^N U_j \phi_j$ with the help of the basis functions and to insert this representation into (4.6) to obtain a discrete system of equations for the coefficient vector $(U_i)_i^N \in \mathbb{R}^N$. If $a(\cdot, \cdot)$ is *not* linear in the first slot —think of nonlinear PDEs such as the Navier-Stokes equations—, then the resulting system is truly nonlinear. But for weak formulation one *linear* PDEs, the linearity in the first slot of $a(\cdot, \cdot)$ allows us to to cast (4.6) into a linear system

$$AU = b \tag{4.7}$$

with $A \in \mathbb{R}^{N \times N}$ and $b \in \mathbb{R}^N$ since

$$a(u_h, \phi_h) = a\Big(\sum_{j=1}^N U_j \phi_j, \phi_h\Big) = \sum_{j=1}^N \underbrace{a(\phi_j, \phi_i)}_{=:A_{ij}} U_j = \underbrace{l(\phi_i)}_{=:b_i} \quad i = 1, \dots N. \tag{4.8}$$

4.1.2 Abstract error theory

Lemma 3 (Galerkin orthogonality)

Assume that $V_h \subset V$ and that $u \in V$ and $u_h \in V_h$ solve the continuous weak formulation (4.4) and the discrete weak formulation (4.5), respectively. Then the error $u - u_h$ satisfies the orthogonality relation

$$a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h. \tag{4.9}$$

Proof. If $v_h \in V_h \subset V$, then the continuous u and the discrete solution u_h satisfy

$$a(u, v_h) = l(v_h),$$
$$a(u_h, v_h) = l(v_h).$$

Subtracting the second equality from the first yields (4.9).

Lemma 4 (Cea's lemma)

Assume that

- $V_h \subset V$
- the continuous weak formulation (4.4) satisfies the assumptions of the Lax-Milgram theorem.
- $u \in V$ is solution to the continuous weak formulation (4.4)
- $u_h \in V_h$ is solution to the discrete weak formulation (4.5)

Then u_h satisfies a quasi best approximation property in the sense that

$$\|u - u_h\| \leqslant \frac{C_a}{\alpha} \inf_{v \in V_h} \|u - v_h\|. \tag{4.10}$$

holds for the error $u - u_h$. Here C_a and α are the boundedness and ellipticity constants for $a(\cdot,\cdot)$ appearing the assumptions for the Lax-Milgram theorem.

Proof. Let $v_h \in V_h$ be fixed but arbitrary, then we wish to show that

$$\|u - u_h\| \leqslant \frac{C_a}{\alpha} \|u - v_h\|. \tag{4.11}$$

The proof of this inequality is rather short. Its main essence consists of three estimates where we first use coercivity of $a(\cdot,\cdot)$ to relate $\|u - u_h\|$ to $a(\cdot,\cdot)$, then add and substract the given v_h and apply Galerkin orthogonality, and finally boundedness of $a(\cdot,\cdot)$ is exploited to estimate the resulting expression. To this end, we see that

$$\begin{aligned}
\alpha\|u - u_h\|^2 &\leqslant a(u - u_h, u - u_h) \\
&= a(u - u_h, u - v_h + v_h - u_h) \\
&= a(u - u_h, u - v_h) + \underbrace{a(u - u_h, v_h - u_h)}_{=0} \\
&\leqslant C_a \|u - u_h\| \|u - v_h\|
\end{aligned} \tag{4.12}$$

Assuming that $\|u - u_h\| \neq 0$ (otherwise (4.11) is trivially satisfied), we can divide (4.12) by $\|u - u_h\|$ and α to see that

$$\|u - u_h\| \leqslant \frac{C}{\alpha} \|u - v_h\|.$$

A primer on finite element methods

Contents

- Galerkin’s method: A recipe to discretize partial differential equations

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The general recipe

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> **Note**
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> The following recipe is deliberately kept vague. Details such as boundary conditions, suitable function spaces, and

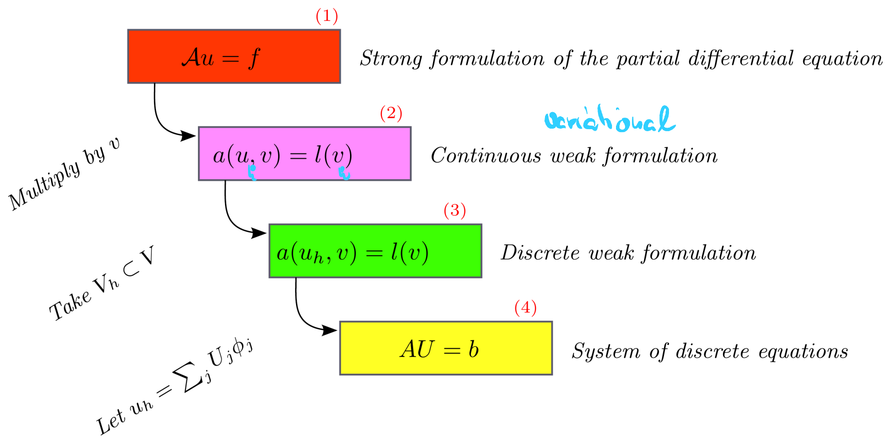


Fig. 1 The four stages of Galerkin’s method to discretize partial differential equations.

Stage 1

Strong formulation of the PDE

Starting point is a partial differential equation

$$\mathcal{A}u = f \qquad (22)$$

in *strong form*: for given function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and partial differential operator $\mathcal{A}$, we assume that the function $u : \Omega \to \mathbb{R}$ satisfies the relation (22) *pointwise* so that $\mathcal{A}u(x) = f(x)\ \forall x \in \Omega$.

Our favorite example is of course the Poisson problem with *homogeneous Dirichlet boundary condition*:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{23}$$

Stage 2

Continuous weak formulation of the PDE

Find $u \in V$ such that

$$a(u, v) = l(v) \quad \forall v \in V. \tag{24}$$

The standard approach to obtain a weak formulation is to multiply with the strong form of the PDE with appropriate test functions which satisfy appropriate smoothness assumptions and —if required— boundary conditions.

For the Poisson problem with homogeneous Dirichlet b.c. we saw previously that we arrived at the weak formulation: Find $u \in V := H_0^1(\Omega)$ s.t. $\forall v \in V$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{l(v)}.$$

Stage 3

Discrete weak formulation of the PDE

By choosing a suitable approximation space $V_h \subset V$ with finite dimension $N = \dim V_h$, we obtain the discrete weak formulation:

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h. \tag{25}$$

Stage 4

Formulation as system of discrete equations

To translate the now finite-dimensional problem (25) into a discrete system of equations, we choose a basis for the discrete function space

$$V_h = \operatorname{span}\{\phi_i\}_{i=1}^N.$$

First observe that the problem (25) is then equivalent to seek a $u_h \in V_h$ such that

$$a(u_h, \phi_i) = l(\phi_i) \quad i = 1, \dots N, \tag{26}$$

since $a(\cdot,\cdot)$ and $l(\cdot)$ are linear in their respectively second and first slot.

The second step is now to rewrite $u_h = \sum_{j=1}^N U_j \phi_j$ with the help of the basis functions and to insert this representation into (26) to obtain a discrete system of equations for the coefficient vector $(U_i)_i^N \in \mathbb{R}^N$. If $a(\cdot,\cdot)$ is *not* linear in the first slot —think of nonlinear PDEs such as the Navier-Stokes equations—, then the resulting system is truly nonlinear. But for weak formulation one *linear* PDEs, the linearity in the first slot of $a(\cdot,\cdot)$ allows us to to cast (26) into a linear system

$$AU = b \tag{27}$$

with $A \in \mathbb{R}^{N\times N}$ and $b \in \mathbb{R}^N$ since

$$a(u_h, \phi_h) = a\Big(\sum_{j=1}^N U_j\phi_j, \phi_h\Big) = \sum_{j=1}^N \underbrace{a(\phi_j, \phi_i)}_{=:A_{ij}} U_j = \underbrace{l(\phi_i)}_{=:b_i} \quad i = 1, \dots N. \tag{28}$$

Abstract error theory

Lemma 3 (Galerkin orthogonality)

Assume that $V_h \subset V$ and that $u \in V$ and $u_h \in V_h$ solve the continuous weak formulation (24) and the discrete weak formulation (25), respectively. Then the error $u - u_h$ satisfies the orthogonality relation

$$a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h. \tag{29}$$

Proof. If $v_h \in V_h \subset V$, then the continuous u and the discrete solution u_h satisfy

$$\begin{aligned} a(u, v_h) &= l(v_h), \\ a(u_h, v_h) &= l(v_h). \end{aligned}$$

Subtracting the second equality from the first yields (29).

Lemma 4 (Cea's lemma)

Assume that

- $V_h \subset V$
- the continuous weak formulation (24) satisfies the assumptions of the Lax-Milgram theorem.
- $u \in V$ is solution to the continuous weak formulation (24)
- $u_h \in V_h$ is solution to the discrete weak formulation (25)

Then u_h satisfies a quasi best approximation property in the sense that

$$\|u - u_h\| \leqslant \frac{C_a}{\alpha} \inf_{v \in V_h} \|u - v_h\|. \tag{30}$$

holds for the error $u - u_h$. Here C_a and α are the boundedness and ellipticity constants for $a(\cdot,\cdot)$ appearing the assumptions for the Lax-Milgram theorem.

Proof. Let $v_h \in V_h$ be fixed but arbitrary, then we wish to show that

$$\|u - u_h\| \leqslant \frac{C_a}{\alpha} \|u - v_h\|. \tag{31}$$

The proof of this inequality is rather short. Its main essence consists of three estimates where we first use coercivity of $a(\cdot,\cdot)$ to relate $\|u - u_h\|$ to $a(\cdot,\cdot)$, then add and substract the given v_h and apply Galerkin orthogonality, and finally boundedness of $a(\cdot,\cdot)$ is exploited to estimate the resulting expression. To this end, we see that

$$\begin{aligned} \alpha\|u - u_h\|^2 &\leqslant a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h + v_h - u_h) \\ &= a(u - u_h, u - v_h) + \underbrace{a(u - u_h, v_h - u_h)}_{=0} \\ &\leqslant C_a \|u - u_h\| \|u - v_h\| \end{aligned} \tag{32}$$

Assuming that $\|u - u_h\| \neq 0$ (otherwise (31) is trivially satisfied), we can divide (32) by $\|u - u_h\|$ and α to see that

$$\|u - u_h\| \leqslant \frac{C}{\alpha} \|u - v_h\|.$$

By André Massing

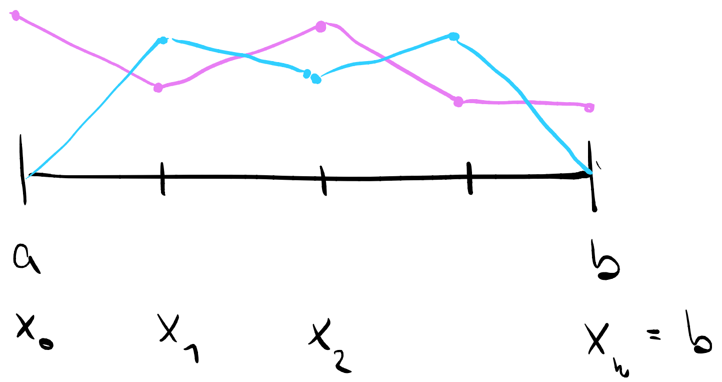
Finite Element : 1d, continuous first order elements

1D: $\Omega = [a,b]$

$$V_h := \{ v \in C([a,b]) : v|_{T_i} \in \hat{P}(T_i) \quad i \in \{1, \dots, n\} \}$$

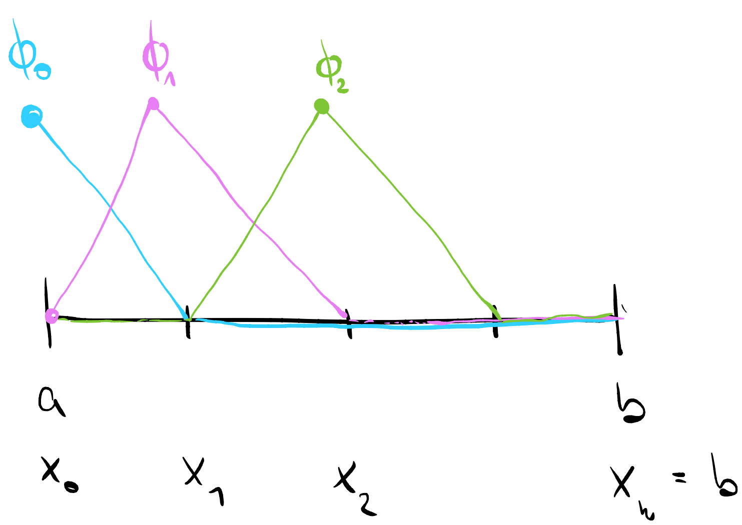
- $T_i = [x_{i-1}, x_i] \quad i = 1, \dots, n \quad ; \quad \{T_i\}_{i=1}^{n} =: \mathcal{T}_h$

$$V_{h,0} = \{ v \in V_h : v|_a = v|_b = 0 \}$$



- Basis functions $\{\phi_i\}_{i=0}^{n}$ for V_h s.t.

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

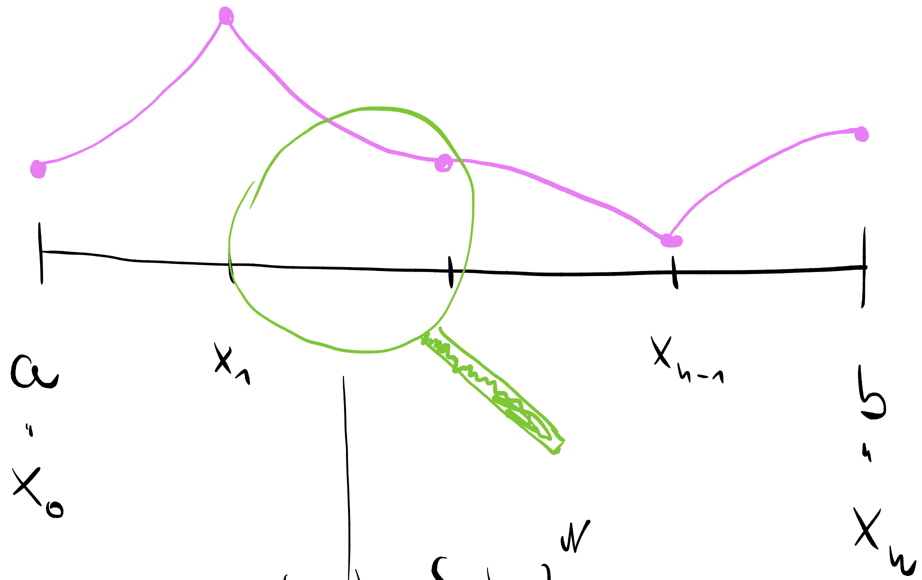


- $\mathcal{I}_h : H^{k+1}(\Omega) \longrightarrow V_h$
- $(\mathcal{I}_h u)(x) = \sum_{i=0}^{n} u(x_i) \cdot \phi_i(x)$

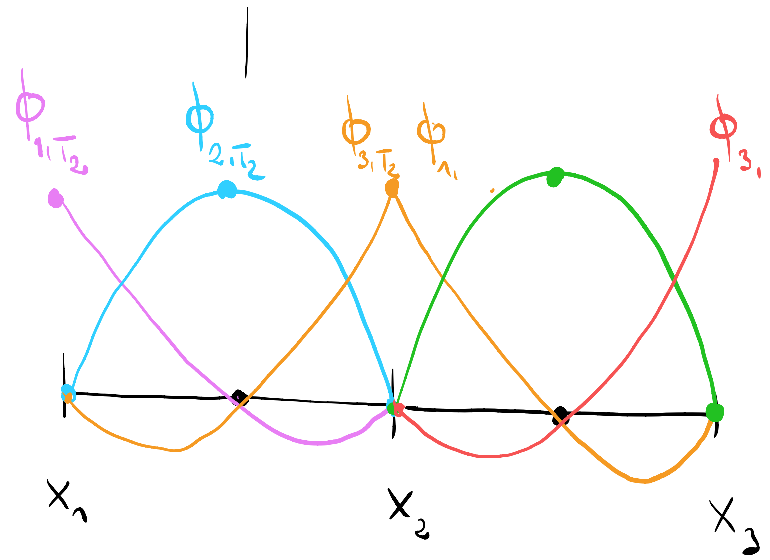
$$\mathcal{I}_h(u)(x_j) = u(x_j).$$

Example : 1d, $\mathbb{P}_c^2$

- $T_i = [x_{i-1}, x_i]$, $i \in \{1, .. n\}$, $\{T_i\}_{i=1}^{n} = \mathcal{J}_h$
- $V_h = \mathbb{P}_c^2(\mathcal{J}_h) = \{ v \in C(\Omega) \mid v|_T \in \mathbb{P}^2(T) \ \forall T \in \mathcal{J}_h \}$



- How do we construct $\{\phi_i\}_{i=1}^{N}$



Interpolation estimates

$$\mathcal{I}_h : H^{k+1}(\Omega) \longrightarrow V_h = \mathbb{P}_c^k(\mathcal{T}_h)$$

- $v \in H^{k+1}(\Omega)$, $h_T = \operatorname{diam}(T)$

$$|v - \mathcal{I}_h v|_{\ell,T} \lesssim h_T^{k+1-\ell} \, |v|_{k+1,T}$$

↓ hidden constant which does not dependent on h.

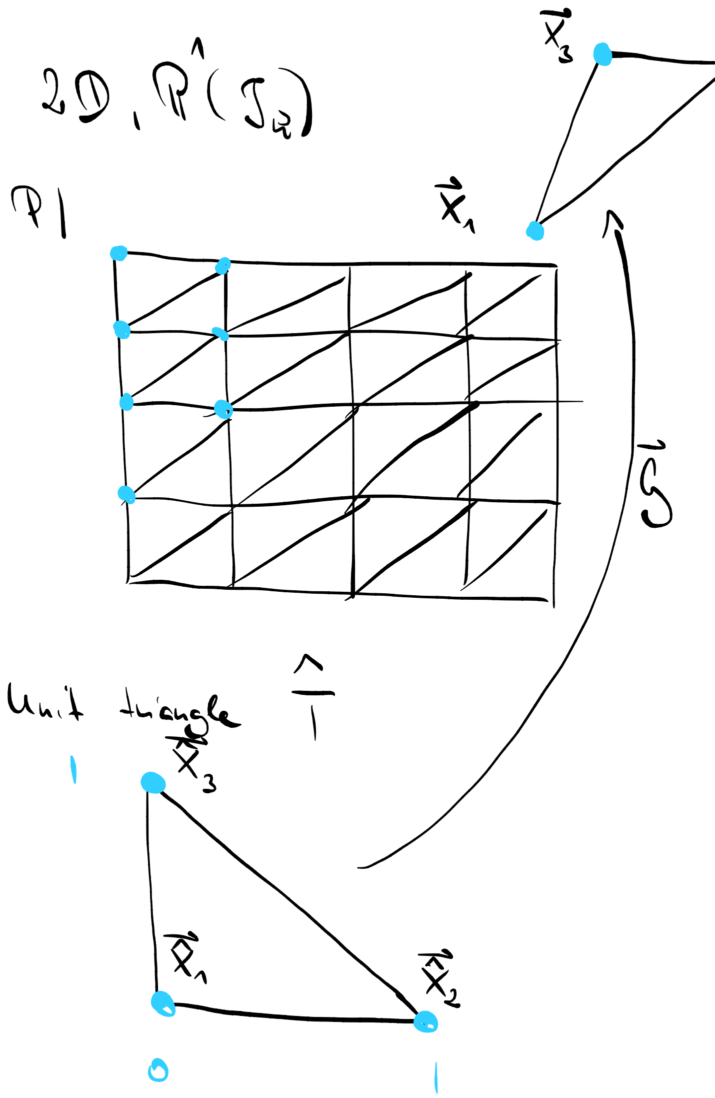
- $h = \max h_T$

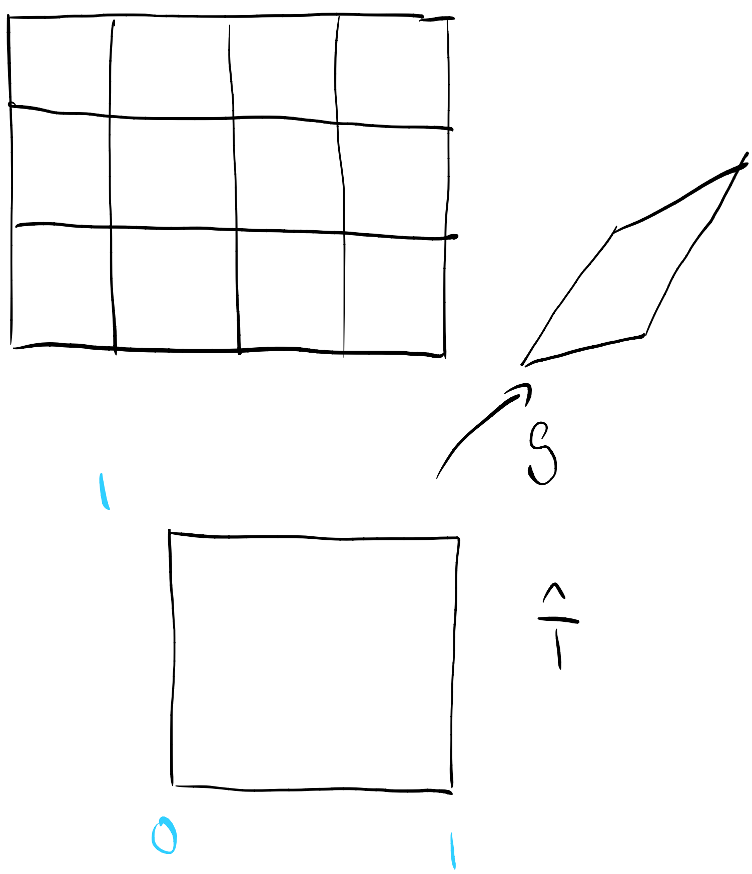
$$|v - \mathcal{I}_h v|_{\ell,\Omega} \lesssim h^{k+1-\ell} \, |v|_{k+1,T} .$$

- $H^s(\Omega)$, $\mathbb{P}^k(\mathcal{T}_h)$ $r = \min\{s, k+1\}$

$$|v - \mathcal{I}_h v|_{\ell,T} \lesssim h^{r-\ell} \, |v|_r .$$

2D, $P^1(T_h)$ $\qquad$ $Q^1(T_h)$





$$\vec{X} = A \cdot \hat{\vec{x}} + \vec{b} = G(\hat{x})$$

$$\vec{X}_i = G(\hat{\vec{x}}_i)$$

$$Q^1(\hat{T}) = \{a + bx + cy + dxy\}$$

$$= P^1([0,1]) \otimes P^1([0,1])$$

$$\phi_i(x) = \hat{\phi}_i \circ G^{-1}(x)$$

$$\hat{\phi}_2(\hat{x}, \hat{y}) = x$$

$$\hat{\phi}_3(\hat{x}, \hat{y}) = \hat{y}$$

$$\hat{\phi}_1(\hat{x}, \hat{y}) = 1 - \hat{x} - \hat{y}$$

• Error estimate for finite element discretizations?

Poisson case we have Cea's lemma: $V_h = \mathbb{P}^k(\mathcal{T}_h)$

$$\|u-u_h\|_{H^1(\Omega)} = \|u-u_h\|_{L^2(\Omega)} + \|\nabla(u-u_h)\|_{L^2(\Omega)}$$

$$\le \frac{C_a}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|_{H^1(\Omega)}$$

$$\le \frac{C}{\alpha} \|u - \mathcal{I}_h\|_{H^1(\Omega)}$$

$$\le \frac{C}{\alpha} h^k |u|_{k+1,\Omega}$$

• Aubin-Nitsche trick lemma:

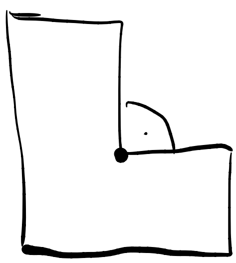
$$\|u-u_h\|_{L^2(\Omega)} \lesssim h^{k+1} |u|_{k+1,\Omega}$$

if $\partial\Omega$ is regular enough so that the "shift-theorem" holds for arbitrary $\psi \in L^2(\Omega)$, i.e.

$$-\Delta\phi = \psi \Rightarrow \phi \in H^2(\Omega).$$

That holds: $\partial\Omega$ is C^∞, convex.

Doesn't hold



TMA4183 - Review on weak formulations of PDEs and finite element methods

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