Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

PDE-constrained control, lecture 6
– Numerical approximation of control problems

Dietmar Hömberg

Outline

- A steepest descent method in function spaces
- Error estimates for the FE-discretization of linear-quadratic elliptic control problems
- A primal dual active set strategy
- SQP methods $\longrightarrow$ see Lecture 7

(CP)

$$\min J(y,u) := \int_\Omega \varphi(x, y(x))dx + \int_\Omega \psi(x, u(x))dx$$

with constraints

$$(\mathrm{SE}) \quad \begin{cases} -\Delta y + y + d(x,y) & = u \quad \text{in } \Omega \\ \dfrac{\partial y}{\partial \nu} & = 0 \quad \text{on } \partial\Omega = \Gamma \end{cases}$$

and $\xi_0(x) \leq u(x) \leq \xi_1(x)$ a.e. in Ω

- $U_{ad} = \{u \in L^\infty(\Omega) \big| \, \xi_0(x) \leq u(x) \leq \xi_1(x)$ a.e. in $\Omega\}$ and $\xi_0, \xi_1 \in L^\infty(\Omega)$.

Projected gradient I

Assume we have already computed $u_1, \ldots u_n$, then to compute u_{n+1} we proceed as follows:

(S1) Compute $y_n = y(u_n)$, solution to (SE)

(S2) Compute p_n as solution to

$$-\Delta p + p + d_y(x, y_n)p = \varphi_y(x, y_n)$$

$$\frac{\partial p}{\partial \nu} = 0\,.$$

(S3) Compute descent direction

$$v_n = -(\psi_n(.,u_n) + p_n)\,.$$

(S4) Compute step-size

$$s_n = \arg\min f(u_n + s v_n)$$

(S5) $u_{n+1} = \mathcal{P}_{[\xi_0,\xi_1]}(u_n + s_n v_n)\,,\ n \mapsto n+1$ Goto (S1)

Projected gradient II

Remark:

- Compared to the linear quadratic case, two difficulties arise:
- computation of y_n requires iterative solver, e.g. Newton's method,
- step-size cannot be computed explicitely, use e.g., Armijo step-size

Linear-quadratic model problem (distributed control)

$$\left.\begin{array}{l} \min J(y,u) = \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\gamma}{2}\|u\|^2_{L^2(\Omega)} \\ \text{bei} \left\{ \begin{array}{rl} -\Delta y = u & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{array} \right\} \qquad \text{(SE)} \\ \text{und} \quad \xi_0 \le u(x) \le \xi_1 \quad \text{a.e. in } \Omega\,. \end{array} \right\} \qquad \text{(CP)}$$

Remarks:

- For $\partial\Omega$ smooth enough, (SE) has a unique solution $y \in H^1_0(\Omega) \cap H^2(\Omega)$ for controls $u \in L^2(\Omega)$
- (CP) has a unique solution $\bar{u}$.

FE discretization of (SE)

- Let $\mathcal{T}_h$ be a regular triangulization Ω, with grid-size $h := \max_{T\in\mathcal{T}_h}\{\mathrm{diam}\,(\mathrm{T})\}$
- $\bar{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} \bar{T}$
- for simplicity assume a polyhedral domain such that $\Omega_h = \Omega$.
- consider piecewise linear finite elements, i.e.

$$V_h = \{v \in C(\bar{\Omega})|v|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h \quad \text{und} \;\; v\Big|_{\partial\Omega} = 0\}$$

  then we have $V_h \subset V = H_0^1(\Omega)$.

- weak formulation for (SE) reads

$$a(y,v) = (\nabla y, \nabla v) = (u,v) \quad \forall v \in V = H_0^1(\Omega)\,. \tag{VE}$$

- Then the fe-solution y_h of (SE) is the unique solution to

$$a(y_h,v) = (u,v) \quad \forall v \in V_h\,. \tag{VE$_h$}$$

Subtracting (VE$_h$) from(VE) gives

$$a(y - y_h, v) = 0 \quad \forall v \in V_h \,. \tag{1}$$

In the so-called energy norm for $v \in V_h$ we obtain

$$\begin{aligned} \|y - y_h\|_E^2 &= a(y - y_h, y - y_h) \overset{(1)}{=} a(y - y_h, y - v) + \underbrace{a(y - y_h, v - y_h)}_{=0} \\ &\le \|y - y_h\|_E \|y - v\|_E \\ &\Rightarrow \|y - y_h\|_E \le \inf_{v \in V_h} \|y - v\| \le \|y - y_h\|_E \,, \text{ hence} \\ \|y - y_h\|_E &= \inf_{v \in V_h} \|y - v\|_E \quad \Leftarrow \text{ Interpolation error .} \end{aligned}$$

FE-Error estimate I – remarks

- The error is optimal in the energy norm.
- Error in solving the FE problem instead of the continuous one is reduced to question of best approximation of sobolev space functions in a FE space.
- FE theory shows error for interpolation is $\|y - I_h y\|_E \leq C(\Omega) h \|y\|_{H^2(\Omega)}$
- hence we obtain

$$\|y - y_h\|_E \leq ch\|y\|_{H^2(\Omega)}$$

L^2-Error estimate – duality argument

The boundary value problem

Discuss Ω and shift index property.

$$
\begin{aligned}
-\Delta w &= y - y_h \quad \text{in } \Omega\\
w &= 0 \qquad\quad \text{on } \partial\Omega
\end{aligned}
$$

has a unique solution $w \in H_0^1(\Omega) \cap H^2(\Omega)$, moreover

$$
\begin{aligned}
\|y - y_h\|^2_{L^2(\Omega)} &= (y - y_h, y - y_h) = a(w, y - y_h)\\
&\overset{(1)}{=} a(w - I_h w, y - y_h) \leq c_1 \|y - y_h\|_E \|w - I_h\|_E\\
&\leq c_2 h \|y - y_h\|_E \underbrace{\|w\|_{H^2(\Omega)}}_{\leq c_3 \|y - y_h\|_{L^2(\Omega)}}
\end{aligned}
$$

$$
\Rightarrow \|y - y_h\|_{L^2(\Omega)} \leq c h^2 \|y\|_{H^2(\Omega)} \,. \tag{2}
$$

L^2-Error estimate for (CP) – I

- Necessary opt. condition for reduced cost functional $f(u) = J(y(u), u)$:

$$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad} = \{u \in L^2(\Omega) | \xi_0(x) \leq u(x) \leq \xi_1(x) \quad \text{a.e. in } \Omega\}$$

$$\Leftrightarrow (S^*(S\bar{u} - y_d) + \gamma\bar{u}, u - \bar{u}) \geq 0 \,.$$

- adjoint p again defined as $p = S^*(S\bar{u} - y_d)$, i.e. p uniquely solves

$$\begin{cases} -\Delta p = \bar{y} - y_d & \text{in } \Omega \\ p = 0 & \text{in } \partial\Omega \,. \end{cases}$$

- we know that $p \in H^2(\Omega) \cap H_0^1(\Omega)$, hence from (2) we infer

$$\|Su - S_h u\|_{L^2(\Omega)} \leq ch^2 \|\widetilde{G}u\|_{H^2(\Omega)} \leq ch^2 \|u\|_{L^2(\Omega)}$$

and since S^* solves the same pde, we have for arbitrary $f \in L^2(\Omega)$

$$\|S^* f - S_h^* f\| \leq ch^2 \|f\|_{L^2(\Omega)} \quad \checkmark \qquad \|S_* \delta\|_{L^2(\Omega)}$$

and also $\quad \|S^* S f - S^* S_h f\|_{L^2(\Omega)} \leq ch^2 \|f\|_{L^2(\Omega)} \,. \quad \checkmark$

L^2-Error estimate for (CP) – II

Now consider semi-discretuzation of (CP):

$$\min \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)} \qquad \text{(CP)}_h$$

$$\text{subject to} \quad a(y_h, v_h) = (u, v_h) \quad \forall v_h \in V_h \quad \text{and} \quad u \in U_{ad}\,.$$

Remark: $(\mathrm{CP})_h$ is called semi-discretization because only the state is discretized.
Define $f_h(u) := J(S_h u, u)$, then

$$f_h'(\bar{u}_h)(u - \bar{u}_h) \geq 0 \quad \forall u \in U_{ad}\,.$$

and thus

$$\big(S_h^*(S_h\bar{u}_h - y_d) + \gamma\bar{u}_h, u - \bar{u}_h\big) \geq 0 \quad \forall u \in U_{ad}\,. \qquad (3)$$

In the continuous case we had

$$\big(S^*(S\bar{u} - y_d) + \gamma\bar{u}, u - \bar{u}\big) \geq 0 \quad \forall u \in U_{ad}\,. \qquad (4)$$

L^2-Error estimate for (CP) – III

Inserting $u = \bar{u}$ in (3) and $u = \bar{u}_h$ in (4) and adding the inequalities yields

$$\big(S^*(S\bar{u} - y_d) - S_h^*(S_h\bar{u}_h - y_d) + \gamma(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u}\big) \geq 0$$

$\Leftrightarrow$

$$\begin{aligned}
\gamma\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} &\leq \big(S^*S\bar{u} - S_h^*S_h\bar{u}_h + (S_h^* - S^*)y_d, \bar{u}_h - \bar{u}\big)\\
&\leq \big(S^*S\bar{u} - S_h^*S_h\bar{u} + S_h^*S_h(\bar{u} - \bar{u}_h) + (S_h^* - S^*)y_d, \bar{u}_h - \bar{u}\big)\\
&\leq \big(\|S^*S\bar{u} - S_h^*S_h\bar{u}\| + \|(S_h^* - S^*)y_d\|\big)\|\bar{u}_h - \bar{u}\|\\
&\qquad + \underbrace{\big(S_h(\bar{u} - \bar{u}_h), S_h(\bar{u}_h - \bar{u})\big)}_{-\|S_h(\bar{u}-\bar{u}_h)\|^2}\\
&\leq \big(c_1 h^2\|\bar{u}\| + c_2 h^2\|y_d\|\big)\|\bar{u}_h - \bar{u}\|\,.
\end{aligned}$$

Theorem

Let $\bar{u}$ the solution to (CP) and $\bar{u}_h$ sol. to $(\mathrm{CP})_h$. then there holds

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq ch^2\Big(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\Big)\,.$$

L^2-Error estimate for (CP) – Remarks

- Numerical solution of (CP) without discretization oc control space possible but tedious, esp. in 3D. Details can be found in
  M. Hinze: A variational discretization concept in control constrained optimization: the linear quadratic case, Comp. Opt. and Anal. 30 (2005), 45–63.
- Another possibility: descritize control space, e.g., piecewise constant, $U_h = \left\{u \in L^2(\Omega) \middle| u_{|T} \equiv const\, \forall T \in \mathcal{T}_k\right\}$
- For completely discretized problem one can show

Theorem

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le ch^2\,.$$

- The proof is more involved, since $\bar{u} \notin U_h$, such that one has to use the projection $\pi_h \bar{u}$ onto U_h.
  see, e.g., C. Meyer, Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints, Control Cybernet., 37 (2008), 51–85.

Active set strategy – continuous case – I

Let S be the solution operator to

$$\begin{cases} -\Delta y &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{auf } \partial\Omega \end{cases} .$$

Then, we know that

$$\bar{u}(x) = \mathcal{P}_{[\xi_0(x),\xi_1(x)]}\Big(-\frac{1}{\gamma}\, p(x)\Big) . \tag{5}$$

Define

$$\mu = -\Big(\frac{1}{\gamma} p + \bar{u}\Big) = -\frac{1}{\gamma}(p + \gamma\bar{u}) = -\frac{1}{\gamma} f'(\bar{u}) ,$$

In view of (5) we obtain

$$\bar{u}(x) = \begin{cases} \xi_0(x) , & \text{if } -\frac{1}{\gamma} p < \xi_0(x) \quad \Leftrightarrow \mu(x) < 0 \\ -\frac{1}{\gamma}\, p(x) , & \text{if } -\frac{1}{\gamma}\, p \in [\xi_0, \xi_1] \quad \Leftrightarrow \mu(x) = 0 \\ \xi_1(x) , & \text{if } -\frac{1}{\gamma}\, p > \xi_1 \quad \Leftrightarrow \mu(x) > 0 \end{cases} ,$$

Active set strategy – continuous case – II

In other words, $u = \bar{u}$ satisfies

$$u(x) = \begin{cases} \xi_0(x)\,, & \text{if } u(x) + \mu(x) < \xi_0(x) \\ -\frac{1}{\gamma}\, p(x)\,, & \text{if } u(x) + \mu(x) \in [\xi_0(x), \xi_1(x)] \\ \xi_1(x)\,, & \text{if } u(x) + \mu(x) > \xi_1(x) \end{cases} \quad . \tag{6}$$

Lemma

$u \in U_{ad}$ *then there holds*

$$u = \mathcal{P}_{[\xi_0(x),\xi_1(x)]}\Big(-\frac{1}{\gamma}\, p(x)\Big) \Leftrightarrow u \textit{ satisfies } (6)\,.$$

Remark: $u + \mu$ may serve as an indicator for the activity of the inequality constraints.

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Optimization II
– Numerical approximation of control problems

Dietmar Hömberg

Topics :

- Error estimate for a semi-discretized OCP
- The primal dual active set method to solve OCP with box constraints.

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Outline

- Error estimates for the FE-discretization of linear-quadratic elliptic control problems
- A primal dual active set strategy

Linear-quadratic model problem (distributed control)

$$\left.\begin{aligned}
&\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)} \\
&\text{bei} \left\{ \begin{aligned} \overset{w_0}{-\Delta y} &= \overset{\beta\downarrow\ ?}{u} && \text{in } \Omega \\ y &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \qquad \text{(SE)} \\
&\text{und} \quad \xi_0 \le u(x) \le \xi_1 \quad \text{a.e. in } \Omega\,.
\end{aligned}\right\} \qquad \text{(CP)}$$

Remarks:

- For $\partial\Omega$ smooth enough, (SE) has a unique solution $y \in H^1_0(\Omega) \cap H^2(\Omega)$ for controls $u \in L^2(\Omega)$
- (CP) has a unique solution $\bar{u}$.

FE discretization of (SE)

- Let $\mathcal{T}_h$ be a regular triangulization Ω, with grid-size $h := \max_{T\in\mathcal{T}_h}\{\mathrm{diam}\,(\mathrm{T})\}$
- $\bar{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} \bar{T}$
- for simplicity assume a polyhedral domain such that $\Omega_h = \Omega$.
- consider piecewise linear finite elements, i.e.

$$V_h = \{v \in C(\bar{\Omega}) | v|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h \quad \text{und} \;\; v\Big|_{\partial\Omega} = 0\}$$

  then we have $V_h \subset V = H_0^1(\Omega)$.

- weak formulation for (SE) reads

$$a(y,v) = (\nabla y, \nabla v) = (u,v) \quad \forall v \in V = H_0^1(\Omega)\,. \tag{VE}$$

- Then the fe-solution y_h of (SE) is the unique solution to

$$a(y_h, v) = (u,v) \quad \forall v \in V_h\,. \tag*{(VE)$_h$}$$

FE-Error estimate I – energy norm

Subtracting (VE$_h$) from(VE) gives

$$a(y - y_h, v) = 0 \quad \forall v \in V_h\,. \tag{1}$$

In the so-called energy norm for $v \in V_h$ we obtain

$$\begin{aligned}
\|y - y_h\|_E^2 &= a(y - y_h, y - y_h) \overset{(1)}{=} a(y - y_h, y - v) + \underbrace{a(y - y_h, v - y_h)}_{=0} \\
&\leq \|y - y_h\|_E \|y - v\|_E \\
&\quad \Rightarrow \|y - y_h\|_E \leq \inf_{v \in V_h} \|y - v\| \leq \|y - y_h\|_E\,, \text{ hence} \\
\|y - y_h\|_E &= \inf_{v \in V_h} \|y - v\|_E \quad \Leftarrow \text{ Interpolation error}\,.
\end{aligned}$$

FE-Error estimate I – remarks

- The error is optimal in the energy norm.
- Error in solving the FE problem instead of the continuous one is reduced to question of best approximation of sobolev space functions in a FE space.
- FE theory shows error for interpolation is $\|y - I_h y\|_E \leq C(\Omega) h \|y\|_{H^2(\Omega)}$
- hence we obtain

$$\|y - y_h\|_E \leq ch\|y\|_{H^2(\Omega)}$$

L^2-Error estimate – duality argument

Aubin-Nitsche argument.

The boundary value problem

$$\begin{aligned} -\Delta w &= y - y_h \quad \text{in } \Omega \\ w &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $w \in H_0^1(\Omega) \cap H^2(\Omega)$, moreover

$$\begin{aligned} \|y - y_h\|^2_{L^2(\Omega)} &= (y - y_h, y - y_h) = a(w, y - y_h) \\ &\overset{(1)}{=} a(w - I_h w, y - y_h) \le c_1 \|y - y_h\|_E \|w - I_h\|_E \\ &\le c_2 h \|y - y_h\|_E \underbrace{\|w\|_{H^2(\Omega)}}_{\le c_3 \|y - y_h\|_{L^2(\Omega)}} \end{aligned}$$

$$\Rightarrow \|y - y_h\|_{L^2(\Omega)} \le ch^2 \|y\|_{H^2(\Omega)} \,. \tag{2}$$

L^2-Error estimate for (CP) – I

Abstract linear quadratic OCP, $J(u) = \frac{1}{2}\|Su - y_d\|_H^2 + \frac{\gamma}{2}\|u\|_U^2$.

- Necessary opt. condition for reduced cost functional $f(u) = J(y(u), u)$:

$$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad} = \{u \in L^2(\Omega) | \xi_0(x) \leq u(x) \leq \xi_1(x) \quad \text{a.e. in } \Omega\}$$

$$\Leftrightarrow (\underbrace{S^*(S\bar{u} - y_d)}_{p} + \gamma\bar{u}, u - \bar{u}) \geq 0\,.$$

- adjoint p again defined as $p = S^*(S\bar{u} - y_d)$, i.e. p uniquely solves

$$\begin{cases} -\Delta p = \bar{y} - y_d & \text{in } \Omega \\ p = 0 & \text{in } \partial\Omega\,. \end{cases}$$

Explain S_h.

- we know that $p \in H^2(\Omega) \cap H_0^1(\Omega)$, hence from (2) we infer

$$\|Su - \boxed{S_h} u\| \leq ch^2 \|\overset{y(u)}{Gu}\|_{H^2(\Omega)} \leq ch^2 \|u\|_{L^2(\Omega)}$$

and since S^* solves the same pde, we have for arbitrary $f \in L^2(\Omega)$

$$\|S^* f - S_h^* f\| \leq ch^2 \|f\|_{L^2(\Omega)}$$

and also $\quad \|S^* S f - S^* S_h f\|_{L^2(\Omega)} \leq ch^2 \|f\|_{L^2(\Omega)}\,.$

L^2-Error estimate for (CP) – II

Now consider semi-discretuzation of (CP):

$$\min \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)} \qquad \text{(CP)}_h$$

$$\text{subject to} \quad a(y_h, v_h) = (u, v_h) \quad \forall v_h \in V_h \quad \text{and} \quad u \in U_{ad}\,.$$

Remark: $(\mathrm{CP})_h$ is called semi-discretization because only the state is discretized.

Define $f_h(u) := J(S_h u, u)$, then

the continuous control which solves the (P_h)

$$f_h'(\bar{u}_h)(u - \bar{u}_h) \geq 0 \quad \forall u \in U_{ad}\,.$$

and thus

Set $u = \bar{u}$

Discrete case

$$\big(S_h^*(S_h\bar{u}_h - y_d) + \gamma\bar{u}_h, u - \bar{u}_h\big) \geq 0 \quad \forall u \in U_{ad}\,. \qquad (3)$$

In the continuous case we had

Set $u = \bar{u}_h$

Continuous

$$\big(S^*(S\bar{u} - y_d) + \gamma\bar{u}, u - \bar{u}\big) \geq 0 \quad \forall u \in U_{ad}\,. \qquad (4)$$

L^2-Error estimate for (CP) – III

Inserting $u = \bar u$ in (3) and $u = \bar u_h$ in (4) and adding the inequalities yields

$$\big(S^*(S\bar u - y_d) - S_h^*(S_h\bar u_h - y_d) + \gamma(\bar u - \bar u_h), \bar u_h - \bar u\big) \geq 0$$

$\Leftrightarrow$

$$
\begin{aligned}
\gamma\|\bar u - \bar u_h\|^2_{L^2(\Omega)} &\leq \big(S^*S\bar u - S_h^*S_h\bar u_h + (S_h^* - S^*)y_d, \bar u_h - \bar u\big)\\
&\leq \big(S^*S\bar u - S_h^*S_h\bar u + S_h^*S_h(\bar u - \bar u_h) + (S_h^* - S^*)y_d, \bar u_h - \bar u\big)\\
&\leq \big(\|S^*S\bar u - S_h^*S_h\bar u\| + \|(S_h^* - S^*)y_d\|\big)\|\bar u_h - \bar u\|\\
&\qquad + \underbrace{\big(S_h(\bar u - \bar u_h), S_h(\bar u_h - \bar u)\big)}_{-\|S_h(\bar u - \bar u_h)\|^2}\\
&\leq \big(c_1h^2\|\bar u\| + c_2h^2\|y_d\|\big)\|\bar u_h - \bar u\|\,.
\end{aligned}
$$

γ denom
very small ...

Theorem

Let $\bar u$ the solution to (CP) and $\bar u_h$ sol. to $(\mathrm{CP})_h$. then there holds

depends on γ^{-1} D

$$\|\bar u - \bar u_h\|_{L^2(\Omega)} \leq ch^2\Big(\|\bar u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\Big)\,.$$

L^2-Error estimate for (CP) – Remarks

- Numerical solution of (CP) without discretization o~~c~~f control space possible but tedious, esp. in 3D. Details can be found in
  M. Hinze: A variational discretization concept in control constrained optimization: the linear quadratic case, Comp. Opt. and Anal. 30 (2005), 45–63.
- Another possibility: descritize control space, e.g., piecewise constant, $U_h = \{u \in L^2(\Omega) \big| u_{|T} \equiv const \, \forall T \in \mathcal{T}_k\}$
- For completely discretized problem one can show

Theorem

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq ch^2 .$$

- The proof is more involved, since $\bar{u} \notin U_h$, such that one has to use the projection $\pi_h \bar{u}$ onto U_h.
  see, e.g., C. Meyer, Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints, Control Cybernet., 37 (2008), 51–85.

Active set strategy – continuous case – I

$J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2} + \frac{\gamma}{2}\|u\|^2_{L^2}$

$\xi_0(x) \le u(x) \le \xi_1(x)$

Let S be the solution operator to

$$\begin{cases} -\Delta y &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{auf } \partial\Omega \end{cases} .$$

Then, we know that

$$\bar{u}(x) = \mathcal{P}_{[\xi_0(x),\xi_1(x)]}\Big(-\frac{1}{\gamma} p(x) \Big) . \tag{5}$$

$u + \mu = -\frac{1}{\gamma} p(x)$

Define

$$\mu = -\Big(\frac{1}{\gamma} p + \bar{u}\Big) = -\frac{1}{\gamma}(p + \gamma\bar{u}) = -\frac{1}{\gamma} f'(\bar{u}) ,$$

In view of (5) we obtain

$$\bar{u}(x) = \begin{cases} \xi_0(x) , & \text{if } -\frac{1}{\gamma} p < \xi_0(x) \quad \Leftrightarrow \mu(x) < 0 \\ -\frac{1}{\gamma} p(x) , & \text{if } -\frac{1}{\gamma} p \in [\xi_0, \xi_1] \quad \Leftrightarrow \mu(x) = 0 \\ \xi_1(x) , & \text{if } -\frac{1}{\gamma} p > \xi_1 \quad \Leftrightarrow \mu(x) > 0 \end{cases} ,$$

Active set strategy – continuous case – II

In other words, $u = \bar{u}$ satisfies

$$u(x) = \begin{cases} \xi_0(x)\,, & \text{if } \overbrace{u(x) + \mu(x)}^{-\frac{1}{\gamma}p(x)} < \xi_0(x) \\ -\frac{1}{\gamma}\,p(x)\,, & \text{if } u(x) + \mu(x) \in [\xi_0(x), \xi_1(x)] \\ \xi_1(x)\,, & \text{if } u(x) + \mu(x) > \xi_1(x) \end{cases} \quad . \tag{6}$$

Lemma

$u \in U_{ad}$ *then there holds*

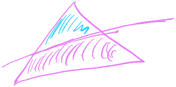
$$u = \mathcal{P}_{[\xi_0(x),\xi_1(x)]}\Big(-\frac{1}{\gamma}\,p(x)\Big) \Leftrightarrow u \textit{ satisfies } (6)\,.$$

Remark: $u + \mu$ may serve as an indicator for the activity of the inequality constraints.

Algorithm: Primal-dual active set strategy

1. Initialize $u_0, \mu_0 \in L^2(\Omega)$, $k := 1$ (u_0 does not have to be admissibe !)
2. Compute

$$\begin{aligned} A_k^+ &= \{x \,|\, u_{k-1}(x) + \mu_{k-1}(x) > \xi_1(x)\} \\ A_k^- &= \{x \,|\, u_{k-1}(x) + \mu_{k-1}(x) < \xi_0(x)\} \\ I_k &= \Omega \setminus (A_k^+ \cup A_k^-)\,. \end{aligned}$$



If $A_k^+ = A_{k-1}^+$ and $A_k^- = A_{k-1}^- \longrightarrow$ terminate, opt. solution reached!

3. solve linear system

State problem

Co-state problem

$$\begin{aligned} -\Delta y &= u\,, && \text{in } \Omega \\ -\Delta p &= y - y_d\,, && \text{in } \Omega \\ y = p &= 0\,, && \text{auf } \partial\Omega \end{aligned} \qquad u = \begin{cases} \xi_0 & \text{auf } A_k^- \\ \xi_1 & \text{auf } A_k^+ \\ -\frac{1}{\gamma}\, p & \text{auf } I_k \end{cases}$$

define

$$u_k := u$$

$$p_k := p$$

$$\mu_k = -\left(\frac{1}{\gamma}\, p_k + u_k\right)$$

$$k \to k+1 \qquad \text{Go to 2.}$$

Reformulation of step 3.

Define characteristic functions for the sets $A_k^{+,-}$:

$$\chi_k^- = \begin{cases} 1\,, & x \in A_k^- \\ 0\,, & \text{else} \end{cases}$$

$$\chi_k^+ = \begin{cases} 1\,, & x \in A_k^+ \\ 0\,, & \text{else} \end{cases}$$

then we can write

$$u + (1 - \chi_k^- - \chi_k^+)\frac{1}{\gamma}\,p = \xi_0 \chi_k^- + \xi_1 \chi_k^+$$

in other words, in step 3. we have to solve

$$\begin{aligned} -\Delta y - u &= 0 \\ -\Delta p - y &= -y_d \\ (1 - \chi_k^- - \chi_k^+)\frac{1}{\gamma}\,p + u &= \xi_0 \chi_k^- + \xi_1 \chi_k^+\,. \end{aligned}$$

Active set strategy – finite dimensional case – I

Let $\bar{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} \bar{T}$ with triangulation $\mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h}\{\mathrm{diam}\,(T)\}$ with m elements. Let u be piecewise constant, i.e.

$U_h = \text{span}\{e_i\}$

$$u = \sum_{i=1}^{m} u_i e_i(x) \quad \text{mit } e_i(x) = \begin{cases} 1\,, & i \in T_i \\ 0\,, & \text{sonst} \end{cases} .$$

Let

$$V_h = \left\{\varphi \in C(\bar{\Omega}_h) \middle| \; \varphi|_T \quad \text{ist linear,} \; \varphi\big|_{\partial\Omega} = 0\right\}$$

with nodal basis $\{\varphi_1, \ldots \varphi_n\}$ for interior nodes $x_1, \ldots x_n$, i.e.,

$$y = \sum_{i=1}^{n} y_i \varphi_i \quad \text{und} \;\; \vec{y} = (y_1, \ldots, y_n)^T$$

Recall state equation

$$a(y,\varphi) = \int_\Omega u\varphi dx$$

Active set strategy – finite dimensional case – II

Inserting y and test function $\varphi = \varphi_i$ vields

$$a(\Sigma y_j\varphi_j,\varphi_i) = \sum_{i=1}^{n} \overset{A\vec{y}}{\underbrace{a(\varphi_i,\varphi_j)}_{\overset{A}{\cancel{K}}_{ij}}}\, y_j = \sum_{j=1}^{m} \boxed{u_j} \overbrace{\int_\Omega e_j\varphi_i}^{\boxed{B_{ij}}} = S_{ij}\varphi_i$$

$$\Leftrightarrow \qquad [\overset{A}{\cancel{K}}\vec{y}]_i = [Bu]_i \quad 1\le i\le n\,,\ K\in\mathbb{R}^{n,n}\,,\ B\in\mathbb{R}^{n,m}$$

hence $\vec{y} = \underset{A^{-1}}{\cancel{K}^{-1}} B\vec{u} = S_h\vec{u}$. $\qquad A\vec{y} = B\vec{u}$

p solves

$$\boxed{a(p,q) = (y-y_d,\varphi)}$$

$$\Leftrightarrow \qquad A\cancel{K}\vec{p} = M\vec{y} - \vec{y}_d \quad \text{with } M_{ij} = \int_\Omega \varphi_i\varphi_j\,dx \text{ and } [\vec{y}_d]_i = \boxed{\int_\Omega y_d\varphi_i}\,dx$$

$$A\vec{p} = M\vec{y} - \vec{y}_d = M(\vec{y} - \vec{z})$$

Active set strategy – discretization of continuous gradient

$f'(y,u)h \quad \nabla f(u)\, h$

$(p + \gamma u, h) \ge 0$, $p \to v_h$, $u \to u_h$, $h \to u_h$

$$
\begin{aligned}
f'h &= \int_\Omega (y-y_d) Sh\,dx + \gamma \int_\Omega uh\,dx = \int_\Omega \underbrace{S^*(y-y_d)}_{p} h\,dx + \gamma \int_\Omega uh\,dx \\
&= \boxed{\int_\Omega ph\,dx + \gamma \int_\Omega uh\,dx} \\
&= \int_\Omega \left(\sum_{i=1}^n p_i\varphi_i\right)\left(\sum_{i=1}^m h_i e_i\right) dx + \overbrace{\gamma \int_\Omega \left(\sum_{i=1}^n u_i e_i\right)\left(\sum_{i=1}^m h_i e_i\right) dx} \\
&= \sum_{j=1}^m \int_{T_j} \left(\sum_{i=1}^n p_i \varphi_i h_j\right) dx + \gamma \sum_{j=1}^m \int_{T_j} \left(\sum_{i=1}^n u_i e_i\right)\left(\sum_{i=1}^m h_i e_i\right) dx \\
&= \sum_{j=1}^m \sum_{i=1}^n p_i \overbrace{\boxed{\int_{T_j} e_j\, \varphi_i}}^{B_{ij}} h_j + \gamma \sum_{j=1}^m u_j h_j |T_j| = \sum_{j=1}^n \left[\left(\sum_{i=1}^n B^T_{ji} p_i\right) + \gamma [Tu]_j\right] h_j
\end{aligned}
$$

$$\gamma \sum_{i,j} u_i h_j \overbrace{\int_\Omega e_i e_j}^{=|T_i|\delta_{ij}}$$

and thus . $f' = B^T \vec{p} + \gamma T \vec{u}$ with T =diag$(|T_1|, \ldots, |T_m|)$

Discrete control problem – I

In comparison, we discretize the continuous control problem and compute the optimality conditions:

$$
\begin{aligned}
J(y,u) &= \frac{1}{2}\int_\Omega (y-y_d)^2 dx + \frac{\gamma}{2}\int_\Omega u^2 dx \\
&= \frac{1}{2}\int_\Omega \left(\sum_{j=1}^{n} y_j\varphi_j - y_d\right)^2 dx + \frac{\gamma}{2}\int_\Omega \left(\sum_{j=1}^{n} u_j e_j\right)^2 \\
&= \frac{1}{2}\int_\Omega \left(\sum_{j=1}^{n} y_j\varphi_j\right)\left(\sum_{j=1}^{n} y_j\varphi_j\right) dx - \int_\Omega \sum_{j=1}^{n} y_j\varphi_j y_d dx \\
&\qquad + \frac{1}{2}\int_\Omega y_d^2 dx + \frac{\gamma}{2}\sum_{j=1}^{m}\int_{T_j} u_j^2 \\
&= \frac{1}{2}\vec{y}^T M\vec{y} - \vec{y}\cdot\vec{y}_d + \frac{\gamma}{2}\vec{u}^T T u + c
\end{aligned}
$$

$$(y_d)_i = \int_\Omega \varphi_j y_d$$

Discrete control problem – II

We obtain

$$\min \frac{1}{2}\,\vec{y}^T M\vec{y} - \vec{y}\cdot\vec{y}_d + \frac{\gamma}{2}\,\vec{u}^T T\vec{u}$$

$$\text{subject to} \qquad \overset{A}{\cancel{K}}\vec{y} = B\vec{u}$$

$$\text{and} \qquad \boxed{\vec{\xi}_0 \le \vec{u} \le \vec{\xi}_1}$$

$$u = \sum_{j=1}^{N_u} u_j e_j$$

$$\xi_0(x) \le u(x) \le \boxed{\xi_1(x)}$$

$$\big\downarrow$$

$$(\xi_0)_j \le u_j \le (\xi_1)_j$$

$$\xi_{1,h} = \Pi_w \cdot u \to u_h$$

Introducing the Lagrangean

$$\mathcal{L}(y,p,u) = \frac{1}{2}\,\vec{y}M\vec{y} - \vec{y}\cdot\vec{y}_d + \frac{\gamma}{2}\,\vec{u}^T T\vec{u} - \langle \underset{A}{\cancel{K}}\vec{y} + B\vec{u}, \vec{p}\rangle$$

the adjoint equation is

$$\mathcal{L}_y = \boxed{M\vec{y} - \vec{y}_d} - \overset{A}{\cancel{K}}\vec{p} \overset{!}{=} 0$$

$$\text{d.h.} \quad \underset{A}{\cancel{K}}\vec{p} = M\vec{y} - \vec{y}_d$$

$$(\xi_1)_j = ?$$

$$\underbrace{(\xi_{1,w}, e_j)_{T_j}} = (\xi_1, e_j)_{T_j}$$

$$\boxed{\xi_{w,j}}\underbrace{(e_j\, T\, e_j)}_{|T|_j} = \int_{T_j} \xi_1$$

and the gradient of the reduced cost functional

$$\mathcal{L}_u(\vec{y},\vec{p},\vec{u}) = \boxed{\gamma T\vec{u} + B^T\vec{p}} = \nabla f(u)$$

$$(\gamma u + p\kappa\sigma)$$

Discrete control problem – III

We obtain the optimality system (cf. lecture 1)

$$K\vec{y} = B\vec{u} \qquad K\vec{p} = M\vec{y} - \vec{y}_d$$

$$\langle \gamma T\vec{u} + B^T p\,,\ \vec{v} - \vec{u}\rangle \geq 0 \quad \forall\ \vec{\xi}_0 \leq \vec{v} \leq \vec{\xi}_1$$

hence, we conclude that the discretized gradient coincides with the gradient of the discretized control problem,

$$\nabla f = 0 \ \Leftrightarrow\ \vec{u} = -\frac{1}{\gamma} T^{-1} B^T \vec{p}.$$

As in the continuous case we define

$$\vec{\mu} = -\left(\frac{1}{\gamma} T^{-1} B^T \vec{p} + \vec{u}\right)$$

and obtain

$$\vec{u}_i = \begin{cases} \xi_0^i\,, & \text{falls } u_i + \mu_i < \xi_0^i \\ \xi_1^i\,, & \text{falls } u_i + \mu_i > \xi_0^i \\ -\dfrac{1}{\gamma}\left[T^{-1} B^T \vec{p}\right], & \text{falls } u_i + \mu_i \in \left[\xi_0^i,\, \xi_1^i\right]. \end{cases}$$

(Discrete) Primal Dual Active Set Strategy

1. Initialize $\vec{u}_0, \vec{\mu}_0 \in \mathbb{R}^m$, $k = 1$

2. Compute

$$A_k^+ = \left\{ i \in \{1, \dots m\} \,\middle|\, u_{k-1,i} + \mu_{k-1,i} > \xi_{1,i} \right\}$$

$$A_k^- = \left\{ i \in \{1, \dots m\} \,\middle|\, u_{k-1,i} + \mu_{k-1,i} < \xi_{0,i} \right\}$$

$$I_k = \{1, \dots, m\} \backslash \{A_k^+ \cup A_k^-\}$$

3. Solve

$$u_i = \begin{cases} \xi_{0,i}\,, & i \in A_k^- \\ \xi_{1,i}\,, & i \in A_k^+ \\ -\frac{1}{\gamma}\left[T^{-1} B^T p\right]_i, & i \in I_k \end{cases} .$$

and define

$$\vec{u}_k = \vec{u} \quad \mu_k = -\left(\frac{1}{\gamma} T^{-1} B^T \vec{p}_k + \vec{u}_k\right)$$

Reformulation of step 3.

Define characteristic functions for the sets $A_k^{+,-}$:

$$\chi^-_{k,ii} = \begin{cases} 1\,, & i \in A_k^- \\ 0\,, & \text{else} \end{cases} \qquad \chi^+_{k,ii} = \begin{cases} 1\,, & i \in A_k^{\cancel{-}\,+} \\ 0\,, & \text{else} \end{cases}$$

then step 3. is equivalent with

$$u + \underbrace{\overbrace{(1-\chi_k^- - \chi_k^+)}^{1}\frac{1}{\gamma}\,T^{-1}}_{E}\,B^T p = \underset{0}{\cancel{\chi_k^-\xi_0 + \chi_k^+\xi_1}}$$

altogether we have to solve

$$\begin{pmatrix} 0 & \overset{A}{\cancel{K}} & -B \\ A\ \cancel{K} & -M & 0 \\ EB^T & 0 & I \end{pmatrix} \begin{pmatrix} \vec{p} \\ \vec{y} \\ \vec{u} \end{pmatrix} = \begin{pmatrix} 0 \\ -\vec{y}_d \\ \chi_k^-\xi_0 + \chi_k^+\xi_1 \end{pmatrix}$$

$=0$ unconstrained problem