TMA4190

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Manifolds
Solutions
Exam June 3, 2008

Problem 1

a) By definition $\overline{A} = \bigcap C$ where the intersection is taken over all C such that $A \subseteq C$ and C is closed in X. Since $\overline{A}$ is the intersection of a family of closed sets, $\overline{A}$ is closed.

b) Let $f(A) \subseteq C$ where C is closed in Y. Then $f^{-1}(C)$ is closed in X since f is continuous, and $A \subseteq f^{-1}(C)$. Thus $\overline{A} \subseteq f^{-1}(C)$, and hence $f(\overline{A}) \subseteq C$. This proves that $f(\overline{A}) \subseteq \overline{f(A)}$.

c) Let $f\colon (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be the identity function on X, i.e. $f(x) = x$ for all $x \in X$. Since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, f is continuous, and then we know that f is a homeomorphism given the compactness and Hausdorff conditions on the topologies. Thus $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 2

We have $((x, y), (h, k) \in \mathbb{R}^m \times \mathbb{R}^n)$

$$f(x + h, y + k) - f(x, y) - [f(x, k) + f(h, y)] = \cdots = f(h, k),$$

and (with the usual norm on $\mathbb{R}^m \times \mathbb{R}^n$ and $\mathbb{R}^p$)

$$\frac{\|f(h,k)\|}{\|(h,k)\|} \leq \frac{\|h\|\|k\|}{\|(h,k)\|} \leq \frac{1}{2}\|(h,k)\|.$$

This shows that $Df(x, y)(h, k) = f(x, k) + f(h, y)$. We have used that $\|(h, k)\| = \sqrt{\|h\|^2 + \|k\|^2}$, and that $2ab \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$.

Problem 3

a) A subset $N \subseteq M$ of a smooth m-manifold M is a submanifold if there is an $n \leq m$ such that for all $p \in N$ there exists a chart (U, φ) on M at p such that

$$\varphi(U \cap N) = \varphi(U) \cap \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n}.$$

b) Here $TN^n \subseteq TM^m$ by considering a curve $\alpha\colon I \to N$ as a curve on M, i.e. $[\alpha]$ maps to $[i \circ \alpha]$ where $i\colon N \to M$ is the inclusion.
Let $[\alpha] \in TN$ and let (U, φ) be a chart on M at $p = \alpha(0) \in N$ such that $\varphi(U \cap N) = \varphi(U) \cap \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R}^{m-n}$. Then we have

the chart $(\pi^{-1}(U), \tilde{\varphi})$ on TM at $[\alpha]$. Here $\tilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$ is given by $\tilde{\varphi}([\beta]) = (\varphi \circ \beta(0), D(\varphi \circ \beta)(0))$. If $[\beta] \in \pi^{-1}(U) \cap TN = \pi^{-1}(U \cap N)$, then $\varphi \circ \beta$ is a curve in $\mathbb{R}^n \times \{0\}$ and thus also $D(\varphi \circ \beta)(0)$ is in $\mathbb{R}^n \times \{0\}$. Hence

$$\tilde{\varphi}(\pi^{-1}(U) \cap TN) = \tilde{\varphi}(\pi^{-1}(U \cap N)) = \varphi(U \cap N) \times (\mathbb{R}^n \times \{0\}) =$$

$$(\varphi(U) \cap \mathbb{R}^n \times \{0\}) \times (\mathbb{R}^n \times \{0\}) \subseteq (\mathbb{R}^n \times \mathbb{R}^{m-n}) \times (\mathbb{R}^n \times \mathbb{R}^{m-n}).$$

By permuting coordinates we see that TN is a submanifold of TM.

c) See the solution of Problem 2c) on the exam from June 6, 2006.

Problem 4

By permuting coordinates if necessary, we may assume that $Df(0) = [\,A \; B\,]$ where A is $m \times m$ and invertible. Define $\tilde{f}\colon U \to \mathbb{R}^m \times \mathbb{R}^{m-n}$ by $\tilde{f}(x, y) = (f(x, y), y)$. Then $\tilde{f}$ is smooth, $f(0, 0) = (0, 0)$, and

$$D\tilde{f}(0,0) = \begin{bmatrix} A & B \\ O & I \end{bmatrix}.$$

Thus $D\tilde{f}(0, 0)$ is invertible, and by the Inverse Function Theorem there exist open sets $W \subseteq U$ and $V \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$, both containing the origin such that $\tilde{f}\colon W \to V$ is a diffeomorphism. Let $\varphi = \tilde{f}^{-1}\colon V \to W$. Then (V, φ) is a chart at 0 in $\mathbb{R}^n$ with $\varphi(V) \subseteq U$, and from $((x, y \in V))$

$$(x, y) = \tilde{f} \circ \varphi(x, y) = (f \circ \varphi(x, y), *)$$

we see that $f \circ \varphi(x, y) = x$ for all $(x, y) \in V$.

Problem 5

a) Since the trace and i, where $i(A) = A^{-1}$, are differentiable, so is f. Let $\mu\colon T_A GL(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ be given by $\mu([\alpha]) = \alpha'(0)$, and define $D_A f\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $D_A f = \mu \circ d_A f \circ \mu^{-1}$. Let α be a curve on $GL(n, \mathbb{R})$ with $\alpha(0) = A$ and $\alpha'(0) = B$. Then

$$D_A f(B) = \mu d_A f([\alpha]) = \mu[f \circ \alpha] = \frac{d}{dt}(\mathrm{tr}(\alpha(t))\alpha(t)^{-1})_{t=0}.$$

Let $\beta(t) = \mathrm{tr}(\alpha(t))\alpha(t)^{-1}$, then $\beta(t)\alpha(t) = \mathrm{tr}(\alpha(t))I$, and

$$\beta'(0)\alpha(0) + \beta(0)\alpha'(0) = \mathrm{tr}(\alpha'(0))I.$$

Thus (with the μ identification) we get

$$d_A f(B) = \beta'(0) = \mathrm{tr}(B)A^{-1} - \mathrm{tr}(A)A^{-1}BA^{-1}.$$

b) If $\mathrm{tr}A = 0$, then $d_A f(B) = \mathrm{tr}(B)A^{-1}$, and $d_A f$ is not onto. Hence A is not a regular point of f. (Note that if $n = 1$, then $\mathrm{tr}A \neq 0$.)