THE INVERSE FUNCTION THEOREM

Let $\mathbb{R}^n$ denote Euclidean n-space with inner product $\langle x, y\rangle = \Sigma x_i y_i$, and norm $\|x\| = \sqrt{\langle x, x\rangle}$. $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the vector space of linear transformations $A\colon \mathbb{R}^n \to \mathbb{R}^m$. We identify $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ with the $m \times n$-matrices, and use the norm given by the inner product $\langle A, B\rangle = \mathrm{tr}(A^\top B)$ (where tr denotes the trace of a matrix). If we further identify $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{nm}$ in the natural way, then this inner product and norm correspond to the Euclidean inner product and norm above.

If $x \in \mathbb{R}^n$ and $A \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$, then $\|Ax\| \leq \|A\|\|x\|$ and if $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, and we think of $yx^\top$ as an element of $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$, then $\|yx^\top\| = \|y\|\|x\|$.

Composition $\circ\colon \mathrm{L}(\mathbb{R}^m, \mathbb{R}^p) \times \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^p)$ is continuous (even smooth). Let $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ denote the subset corresponding to the invertible $n \times n$-matrices. This is an open subset of $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ and the map $\mathrm{inv}\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ given by $\mathrm{inv}(A) = A^{-1}$, is continuous (even smooth) as is seen for example from Cramer's Rule.

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. Recall that a function $f\colon U \to V$ is differentiable at $x_0 \in U$ if there is an element $A \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$(*) \qquad \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - Ah}{\|h\|} = 0.$$

If $(*)$ holds, then $Ax = \lim\limits_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$ $(t \in \mathbb{R})$ for all $x \in \mathbb{R}^n$, and A is unique if it exists. If f is differentiable at x_0, we call A the derivative of f at x_0 and denote it by $Df(x_0)$. It is easy to see that if f is differentiable at x_0, then f is continuous at x_0. (This also follows from the lemma below.) If f is differentiable at all $x \in U$, we call f differentiable in U.

Lemma. *Let $U \subseteq \mathbb{R}^n$ be open, and let $f\colon U \to \mathbb{R}^m$ be a function. Then f is differentiable at $x_0 \in U$ if and only if there exists a function $\alpha\colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that α is continuous at x_0 and $f(x) = f(x_0) + \alpha(x)(x - x_0)$ for all $x \in U$.*

Proof. If f is differentiable at x_0, define $\alpha\colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ by

$$\alpha(x) = \begin{cases} Df(x_0) + \frac{r(x)(x - x_0)^\top}{\|x - x_0\|^2} & \text{if } x \neq x_0 \\ Df(x_0) & \text{if } x = x_0, \end{cases}$$

where $r(x) = f(x) - f(x_0) - Df(x_0)(x - x_0)$, $x \in U$. Then clearly $f(x) = f(x_0) + \alpha(x)(x - x_0)$ for all $x \in U$. From $\|r(x)(x - x_0)^\top\| = \|r(x)\|\|x - x_0\|$ and $(*)$ above, we see that α is continuous at x_0. Next assume that a function α as in the lemma exists. Then the continuity of α at x_0 gives that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - \alpha(x_0)h\|}{\|h\|} = \lim_{h \to 0} \frac{\|[\alpha(x_0 + h) - \alpha(x_0)]h\|}{\|h\|} \leq \lim_{h \to 0} \|\alpha(x_0 + h) - \alpha(x_0)\| = 0.$$

Hence f is differentiable at x_0. $\square$

Note that if a function α as in the lemma exists, then $Df(x_0) = \alpha(x_0)$. If $n = m = 1$ in the lemma, and f is differentiable at x_0, then α is given by

$$\alpha(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0. \end{cases}$$

Theorem (The Chain Rule). *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets, and let $f\colon U \to V$ and $g\colon V \to \mathbb{R}^p$. If f is differentiable at $x_0 \in U$ and g is differentiable at $f(x_0)$, then $g \circ f\colon U \to \mathbb{R}^p$ is differentiable at x_0, and*

$$D(g \circ f)(x_0) = Dg\left(f(x_0)\right) \circ Df(x_0).$$

Proof. By the lemma, there exists $\alpha\colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that α is continuous at x_0 and $f(x) = f(x_0) + \alpha(x)(x - x_0)$ for all $x \in U$; and $\beta\colon V \to \mathrm{L}(\mathbb{R}^m, \mathbb{R}^p)$ continuous at $y_0 = f(x_0)$, such that

$g(y) = g(y_0) + \beta(y)(y - y_0)$ for all $y \in V$. For $x \in U$ we then have

$$(g \circ f)(x) = (g \circ f)(x_0) + [\beta(f(x)) \circ \alpha(x)](x - x_0),$$

and $\gamma \colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^p)$ given by $\gamma(x) = \beta(f(x)) \circ \alpha(x)$ is continuous at x_0 since f and α are continuous at x_0, β is continuous at $f(x_0)$, and composition is continuous. By the lemma, $g \circ f$ is differentiable at x_0 with $D(g \circ f)(x_0) = \beta(f(x_0)) \circ \alpha(x_0) = Dg(f(x_0)) \circ Df(x_0)$. $\square$

We also have that if $id \colon U \to U$ (U open in $\mathbb{R}^n$) denotes the identity function given by $id(x) = x$ for all $x \in U$, then $Did(x_0) = I$ (the identity function $\mathbb{R}^n \to \mathbb{R}^n$) for all x_0. Also an inclusion $i \colon U \to V$ ($U \subseteq V \subseteq \mathbb{R}^n$ open subsets) has $Di(x_0) = I$ for all $x_0 \in U$.

Corollary. *Let $U, V \subseteq \mathbb{R}^n$ be open sets, and let $f \colon U \to V$ be a bijection. If f is differentiable at $x_0 \in U$ and f^{-1} is differentiable at $f(x_0)$, then $Df(x_0)$ is invertible and $Df^{-1}(f(x_0)) = Df(x_0)^{-1}$.* $\square$

Theorem (The Mean Value Theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be open, and let $f \colon U \to \mathbb{R}^m$ be a differentiable in U. If $x, x' \in U$ are such that $(1-t)x + tx' \in U$ for all $t \in [0,1]$ and $\|Df((1-t)x + tx')\| \leq M$ for all $t \in [0,1]$, then $\|f(x') - f(x)\| \leq M\|x' - x\|$.*

Proof. Define $\phi \colon [0,1] \to \mathbb{R}$ by $\phi(t) = \langle f((1-t)x + tx'), f(x') - f(x) \rangle$. Then ϕ is continuous, and differentiable on $(0,1)$ with $\phi'(t) = \langle Df((1-t)x + tx')(x' - x), f(x') - f(x) \rangle$ by the Chain Rule. Hence $\phi(1) - \phi(0) = \phi'(t_0)$ for some $t_0 \in (0,1)$. Let $z = (1-t_0)x + t_0 x'$. Then using the Cauchy–Schwarz inequality ($|\langle x, y \rangle| \leq \|x\|\|y\|$), we get

$$\begin{aligned} \|f(x') - f(x)\|^2 &= \phi(1) - \phi(0) \\ &= \langle Df(z)(x' - x), f(x') - f(x) \rangle \\ &\leq \|Df(z)(x' - x)\| \|f(x') - f(x)\| \end{aligned}$$

and hence $\|f(x') - f(x)\| \leq \|Df(z)\| \|x' - x\| \leq M\|x' - x\|$. $\square$

Proposition. *Let $U, V \subseteq \mathbb{R}^n$ be open sets, and let $f \colon U \to V$ be a bijection. If f is differentiable at $x_0 \in U$ with $Df(x_0)$ invertible and f^{-1} is continuous at $f(x_0)$, then f^{-1} is differentiable at $f(x_0)$.*

Proof. Since f is differentiable at x_0, the lemma gives that there exists a function $\alpha \colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$, continuous at x_0, such that $f(x) = f(x_0) + \alpha(x)(x - x_0)$ for all $x \in U$.

Recall that $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ is open. Then from $\alpha(x_0) = Df(x_0) \in \mathrm{GL}(n, \mathbb{R})$ and the continuity of α at x_0, we get that there exists an open neighborhood $U' \subseteq U$ of x_0 such that $\alpha(x) \in \mathrm{GL}(n, \mathbb{R})$ for all $x \in U'$. Let $g = f^{-1} \colon V \to U$ and let $y_0 = f(x_0) \in V$. Since f^{-1} is assumed to be continuous at y_0, there exists an open neighborhood $V' \subseteq V$ of y_0 such that $V' \subseteq g^{-1}(U') = f(U')$. For $y \in V'$ we have $y - y_0 = f(g(y)) - f(x_0) = \alpha(g(y))(g(y) - x_0)$. Since $y \in V'$, $g(y) \in U'$, and $\alpha(g(y))$ is invertible. Hence we have

$$g(y) = g(y_0) + \alpha(g(y))^{-1}(y - y_0)$$

for all $y \in V'$. Let $\beta \colon V' \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ be given by $\beta(y) = \alpha(g(y))^{-1}$. Then β is continuous at y_0 since g is continuous at y_0, α is continuous at $x_0 = g(y_0)$, and $\mathrm{inv} \colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is continuous. Hence g is differentiable at y_0, i.e. f^{-1} is differentiable at $f(x_0)$. $\square$

Compare this with the corollary above.

Theorem (The Inverse Function Theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be open, and let $f \colon U \to \mathbb{R}^n$ be a C^1-function. If $x_0 \in U$ and $Df(x_0)$ is invertible, then there exists open sets $V \subseteq U$ and $W \subseteq f(U)$ such that $x_0 \in V$ and $f \colon V \to W$ is a C^1-diffeomorphism.*

Proof. Recall that $f \colon U \to R^n$ is a C^1-function if f is differentiable in U and $Df \colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous, and that $f \colon V \to W$ is a C^1-diffeomorphism if f is C^1-function, is invertible, and also f^{-1} is a C^1-function.

We first show that f is injective near x_0. Let $\tilde{f} \colon U \to \mathbb{R}^n$ be given by $\tilde{f}(x) = x - Df(x_0)^{-1} f(x)$. Then $D\tilde{f}(x) = I - Df(x_0)^{-1} Df(x)$ for all $x \in U$, and $\tilde{f}$ is C^1. Since $D\tilde{f}(x_0) = \mathcal{O}$ and $D\tilde{f} \colon U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous, we can find a $\delta > 0$ such that $B_\delta(x_0) \subseteq U$ and $\|D\tilde{f}(x)\| \leq \frac{1}{2}$ for all $x \in B_\delta(x_0)$. [Here $B_r(x_0) = \{x \mid \|x - x_0\| < r\}$ is the open r-ball ($r > 0$) at x_0. The closed r-ball is denoted by $\overline{B}_r(x_0)$.]

Note that $Df(x)$ is invertible for all $x \in B_\delta(x_0)$. If $x, x' \in B_\delta(x_0)$ with $f(x) = f(x')$, then by the Mean Value Theorem, we have $\|x' - x\| = \|\tilde{f}(x') - \tilde{f}(x)\| \leq \frac{1}{2}\|x' - x\|$, and hence $x = x'$. Thus f is injective on $B_\delta(x_0)$.

Let $V = B_\delta(x_0)$ and $W = f(V)$, then $f\colon V \to W$ is a continuous bijection. Next we show that $W \subseteq \mathbb{R}^n$ is open, and that $f^{-1}\colon W \to V$ is continuous.

In order to show that W is open, let $y \in W$, say $y = f(x)$ where $x \in V$. Since V is open, we can find a closed r-ball $\overline{B}_r(x) \subseteq V$. We claim that $B_\epsilon(y) \subseteq W$ where $\epsilon = r/(2\|Df(x_0)^{-1}\|)$. Let $y' \in B_\epsilon(y)$ and define $F\colon \overline{B}_r(x) \to \mathbb{R}^n$ by $F(z) = z + Df(x_0)^{-1}\,(y' - f(z))$. Note that if $F(x') = x'$, then $y' = f(x')$ and $y' \in W$. So we prove that F has a fixed point. First we show that F maps $\overline{B}_r(x)$ to itself. Let $z \in \overline{B}_r(x)$, then

$$\begin{aligned}
\|F(z) - x\| &\leq \|F(z) - F(x)\| + \|F(x) - x\| \\
&= \|\tilde{f}(z) - \tilde{f}(x)\| + \|Df(x_0)^{-1}(y' - y)\| \\
&\leq \tfrac{1}{2}\|z - x\| + \|Df(x_0)^{-1}\|\|y' - y\| \\
&\leq \tfrac{r}{2} + \tfrac{r}{2} \\
&= r,
\end{aligned}$$

and we see that we may view F as a map $F\colon \overline{B}_r(x) \to \overline{B}_r(x)$. But $\overline{B}_r(x)$ is a complete metric space and Banach's Fixed Point Theorem gives that F has a fixed point $x' \in \overline{B}_r(x)$ since F is a contraction: For $z, z' \in \overline{B}_r(x)$ we have (as above)

$$\|F(z) - F(z')\| = \|\tilde{f}(z) - \tilde{f}(z')\| \leq \tfrac{1}{2}\|z - z'\|.$$

Since $x' \in \overline{B}_r(x) \subseteq V$ and $f(x') = y'$, we see that $B_\epsilon(y) \subseteq W$, and W is an open subset of $\mathbb{R}^n$.

The same argument shows that if $V' \subseteq V$ is open, then $f(V')$ is open in $\mathbb{R}^n$ and hence $f(V') \subseteq W$ is open. But then $g = f^{-1}\colon W \to V$ is continuous since if $V' \subseteq V$ is open, then $g^{-1}(V') = f(V') \subseteq W$ is open.

Since $Df(x)$ is invertible for all $x \in V$, and we just proved that $f^{-1}\colon W \to V$ is continuous, we get by the proposition that $f^{-1}\colon W \to V$ is differentiable. Since $Df^{-1}(y) = Df\left(f^{-1}(y)\right)^{-1}$ it follows that $Df^{-1}\colon W \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous, and f^{-1} is C^1. Thus $f\colon V \to W$ is a C^1-diffeomorphism. □

It is easy to extend the Inverse Function Theorem to a statement about C^r-functions ($r \geq 2$) and smooth functions.

Exercise. With the notation of the last proof, show that $\|g(y') - g(y'')\| \leq 2\|Df(x_0)^{-1}\|\|y' - y''\|$ for all $y', y'' \in W$. This gives another proof of the continuity of f^{-1}. □

(I.H., 11. mai 2012)