NTNU

Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4195 Mathematical Modeling**

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Examination time (from–to): 9:00–13:00

Permitted examination support material:
C: Approved simple calculator, Rottman: *Matematisk formelsamling*.

Other information:
There are 8 problems on this exam: 1a) – 1b), 2, 3, 4a) – 4d).
Every problem can be solved independently of the solutions of the other problems.

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Informasjon om trykking av eksamensoppgave
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Date Signature

Problem 1 A population model is given in the following form:

$$\begin{cases} \dot{x} = rx\Big(1 - \dfrac{x}{K}\Big) - \dfrac{axy}{c+x}, \\ \dot{y} = -my + \dfrac{bxy}{c+x}, \end{cases}$$

where $r, m, K, a, b, c > 0$ are constants and $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$.

a) Explain the different terms and quantities in this model. What sort of interaction is there between populations here?

What could these equations model? Give an example.

We now consider a situation where after scaling the model takes the form

$$\begin{cases} \dot{x} = 2x\Big(1 - \dfrac{x}{2}\Big) - \dfrac{xy}{1+x}, \\ \dot{y} = -y + \dfrac{2xy}{1+x}. \end{cases} \tag{1}$$

b) Find all the equilibrium points (x_e, y_e) of (1) such that $x_e, y_e \geq 0$, and then determine their stability.

Hint: The right hand side of the first equation has a common factor x.

Problem 2 A physical system is governed by the boundary value problem

$$y'' + f(\varepsilon y) = 0 \quad \text{for} \quad 0 < x < 1, \qquad y(0) = 0, \quad y(1) = \varepsilon,$$

where f is a smooth function, $0 < \varepsilon < 1$, and $y'' = \frac{\partial^2 y}{\partial x^2}$.

Use perturbation methods to find an approximate solution with $O(\epsilon^2)$ error.

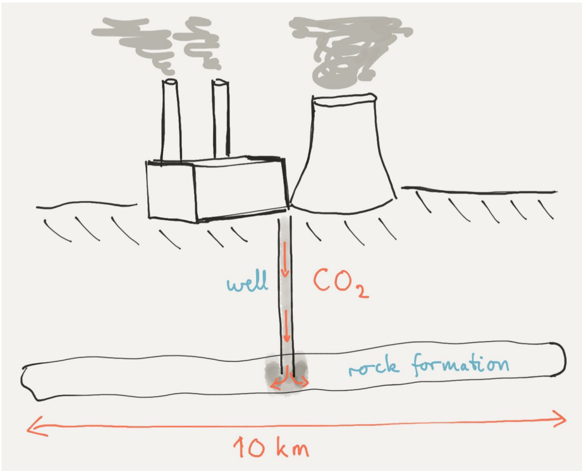
Problem 3 A scaled fluid density ρ satisfies the following initial value problem,

$$\begin{cases} \rho_t + j(\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho(x,0) = \rho_0(x), & t = 0,\ x \in \mathbb{R}, \end{cases}$$

with $j(\rho) = e^\rho$. Find the solution ρ in the following two cases:

$$(i) \quad \rho_0(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases} \qquad \text{and} \qquad (ii) \quad \rho_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Problem 4 In CO_2 sequestration, a porous rock formation is used to store CO_2 produced for example by a coal power plant. The gas is compressed and injected into the formation at a vertical well and will diffuse into the surroundings. The gas cannot escape since the rock formation is surrounded by impermeable rock.



Only the pore volume, a constant fraction $\phi \in (0, 1)$ of the total volume, is available for storage, and the mass flux j^* of CO_2 in the rock formation will be driven by pressure differences according to Darcy's law:

$$j^* = -\rho^* \frac{k}{\mu} \nabla p^*,$$

where ρ^* and μ are the mass density and viscosity of CO_2, k is the permeability of the rock, and ∇p^* is the spatial gradient of the CO_2 pressure p^*.

We will study the case where CO_2 is injected as a gas satisfying the ideal gas law,

$$p^* = \rho^* RT,$$

where R is the gas constant and T is the temperature, which is constant here.

a) Carefully explain how you can use conservation of mass in a shrinking control volume in $\mathbb{R}^3$ (e.g. balls $B(\vec{x}, r) = \{\vec{y} : |\vec{y} - \vec{x}| < r\}$) to show that at any point (x^*, y^*, z^*) inside the rock formation,

$$\phi \rho^*_{t^*} + \nabla \cdot j^* = 0,$$

where $\nabla \cdot j^* = j^*_{x^*} + j^*_{y^*} + j^*_{z^*}$. Then show that

$$\rho^*_{t^*} = \kappa \, \nabla \cdot (\rho^* \nabla \rho^*) = \kappa \Big[(\rho^* \rho^*_{x^*})_{x^*} + (\rho^* \rho^*_{y^*})_{y^*} + (\rho^* \rho^*_{z^*})_{z^*} \Big], \tag{2}$$

and determine the constant κ.

Hint: You may assume that ρ^* and j^* are smooth.

We will now consider a long thin rock formation the shape of a recangular box of length $L = 10\,\text{km}$, height and width $H = W = 500\,\text{m}$, and let

$$\varepsilon = \frac{H}{L} = \frac{W}{L} = 0{,}05.$$

In order not to fracture the rock and reduce the storage capacity of the rock formation, the CO_2 pressure at the injection well is maintained at a moderate level $\bar{p}$. This will be the maximal CO_2 pressure in the rock formation.

b) Explain why $\bar{\rho} = \frac{\bar{p}}{RT}$ is a natural scale for ρ^* in this problem.

Find good scales for x^*, y^*, z^*, and t^* such that after scaling, equation (2) becomes

$$\rho_t = (\rho \rho_x)_x + \frac{1}{\varepsilon^2} \Big[(\rho \rho_y)_y + (\rho \rho_z)_z \Big]. \tag{3}$$

We now assume we are in a situation where $\rho^* = \rho^*(x^*, t^*)$ and the problem is one dimensional. The well is modelled as a point source and equations (2) and (3) reduce to

$$\rho^*_{t^*} = \kappa \, (\rho^* \rho^*_{x^*})_{x^*}, \tag{4}$$

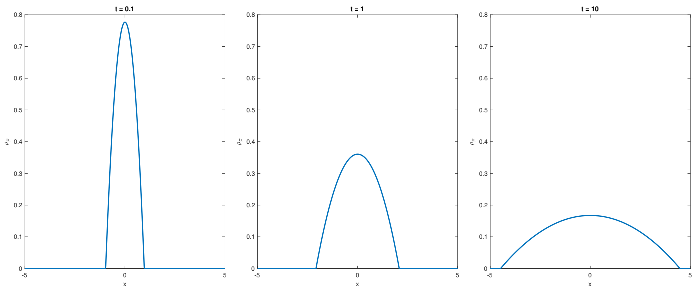
and

$$\rho_t = (\rho \rho_x)_x. \tag{5}$$

To study this case, we introduce a function ρ_F defined by

$$\rho_F(x,t) = \begin{cases} ct^{-\frac{1}{3}} - \frac{1}{6} x^2 t^{-1} & \text{for } x^2 \leq 6ct^{\frac{2}{3}}, \\ 0 & \text{for } x^2 \geq 6ct^{\frac{2}{3}}. \end{cases} \tag{6}$$

where $c = (\frac{3}{32})^{\frac{1}{3}}$ is chosen such that $\int_{-\infty}^{\infty} \rho_F(x,t) dx = 1$. Note that ρ_F is non-negative and continuous, and its shape is plotted below for increasing times:



c) Show that $\rho_F(x,t)$ is a solution of (5) when $t > 0$ and $x^2 \neq s(t)^2$ where

$$s(t) = (6c)^{\frac{1}{2}} t^{\frac{1}{3}}.$$

Show that $\rho_F(x,t)$ is a (weak) solution of (5) also at $x = s(t)$ in the sense that it satisfies the corresponding (scaled) conservation law in integral form on any control volume $[a,b]$ where $0 < a < s(t) < b$.

[*Hint:* In this case the (scaled) flux is $j = -\rho_F(\rho_F)_x$. One possibility is to use the first part of c).]

Let $f(x)$ be continuous and show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} \rho_F(x,t) f(x)\, dx = f(0).$$

Explain why $\rho_F(x,t)$ is a fundamental solution of (5).

After an initial test period of 5 days, the CO_2 injection is halted for 3 months. During the test, $M = 10^8\,\text{kg}$ of CO_2 was injected into the rock formation, and measurements showed that the gas was confined within a distance of $l = 5\,\text{m}$ of the well when the injection stopped.

d) Assume that we are in a situation that can be modelled by equation (4) with $\kappa = 10^{-5}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}}$, and that the well is in the center of the (one dimensional) rock formation.

Use the method of intermediate asymptotics to find an approximation of the gas density ρ^* when 3 months has passed.

Does this approximate solution satisfy the correct boundary conditions?

Hint: Scaling! The results in part c) may be useful. Recall that the rock formation has length $L = 10\,\text{km}$.