NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4195 Mathematical Modelling**

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Examination date: 18th December, 2018
Examination time (from–to): 15:00–19:00
Permitted examination support material:

- Rottmann, Mathematical formulae.
- Approved basic calculator.

Other information:

- You may answer to the questions of the exam either in English or in Norwegian.
- Good luck!

Language: English
Number of pages: 3
Number of pages enclosed: 0

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Informasjon om trykking av eksamensoppgave	
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Date Signature

Problem 1 Consider the initial value problem

$$
\begin{aligned}
m\ddot{y}^* &= -\lambda(\dot{y}^*)^3 - ky^*, \qquad (1)\\
y^*(0) &= 0,\\
\dot{y}^*(0) &= v_0,
\end{aligned}
$$

which models a spring pendulum of mass m, spring constant k, and cubic damping with damping parameter λ, which is perturbed from its equilibrium position with initial velocity $v_0 \neq 0$.

Propose a rescaling for this equation that is valid for times close to zero based on the assumption that the first two terms in (1) (acceleration and damping) dominate. Under which condition does this yield a reasonable scaling?

Problem 2 Consider the differential equation

$$\epsilon y'' + y' = \frac{y + y^3}{1 + 3y^2}$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. Find leading order outer, inner and uniform solutions for small $\epsilon > 0$ using the fact that there is a boundary layer at $x = 0$.

Hint: It is enough to provide an implicit form for the outer solution.

Problem 3 The maximal power P that a wind turbine can produce depends on the density ρ of the air, the air velocity v, and the length r of its rotor blades. The dimensions of these physical quantities are $[P] = \mathrm{kg\,m^2/s^3}$, $[\rho] = \mathrm{kg/m^3}$, $[v] = \mathrm{m/s}$, and $[r] = \mathrm{m}$. Find the most general dimensionally consistent model for the power P depending on the other physical quantities mentioned above.

Wind turbines start to operate only when a minimum output power is reached. For air densities of approximately $\rho_E \approx 1.2\mathrm{kg/m^3}$, which are typical at the surface of the earth at medium temperatures, this happens at wind speeds of about $v_E \approx 4\mathrm{m}/s$.
In contrast, the air density on the surface of Mars is about $\rho_M \approx 2 \cdot 10^{-2}\mathrm{kg/m^3}$, and wind speeds are about 8m/s during a typical day, and about 20m/s during a storm. Would a standard terrestian wind turbine operate during a typical martian day or during a martian storm?

Problem 4 In order to control insect numbers, it has been suggested to maintain a stable number of sterile male insects in the population. The main idea is that the sterile males compete with fertile males over females; however, if females mate with the sterile males, no offspring is produced. This can effectively reduce the reproduction rate of the insects.

We consider now specifically a model where the population of insects is described, after rescaling, by the equation

$$\frac{\partial N}{\partial t} = \frac{N^2}{N+S} - (\kappa + N + S)N$$

for some parameter $0 < \kappa < 1$.

Determine the equilibrium states of the insect population as a function of S, and discuss the stability of these equilibria. What is the smallest non-zero stable population of insects that can be achieved according to this model?

Problem 5 This exercise is concerned with the formulation of a traffic flow model for ant trails: As ants move along an established trail, they deposit a *trail pheromone*, which slowly evaporates over time. This trail pheromone serves as an orientation marker for the ants traveling along the trail.

Derive a PDE model for the ant density $\rho(x,t)$ and pheromone concentration $p(x,t)$ along an infinitely long ant trail that is based on the "conservation of ants along the trail" and the following assumptions:

- As each ant moves, it deposits a trail pheromone at a constant rate.
- The trail pheromone evaporates at a constant rate.
- There is a chance that an ant will lose the trail. The rate at which this happens is a function $f(p)$ depending on the pheromone concentration p.
- The speed of the ants is given by

  $$v = v_0(1 + \alpha p)(\rho_{\max} - \rho)$$

  for ant densities smaller than the maximal density $\rho_{\max}$.

Problem 6 A satellite that orbits the earth satisfies the system of equations

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2},$$
$$\frac{d}{dt}\Big(r^2\dot{\theta}\Big) = 0,$$

where (r, θ) is the position of the satellite in polar coordinates (centered at the center of the earth), G is the gravitational constant, and M the mass of the earth. One possible solution of this system of equations is given by

$$r = a, \qquad \dot{\theta} = \omega, \qquad \text{with} \qquad a^3\omega^2 = GM.$$

This describes a circular orbit at constant radius a and constant angular velocity ω.

We now consider a small perturbation of that circular orbit and use a regular perturbation of the form $r = a + \epsilon r_1 + \dots$ and $\dot{\theta} = \omega + \epsilon\dot{\theta}_1 + \dots$ in order to find a linearisation of this equation around the circular orbit. Find the equations for r_1 and $\dot{\theta}_1$, and verify that the radial term of the linearised solution has the general form

$$r_1(t) = A\sin(\omega t) + B\cos(\omega t) + C$$

and thus remains bounded for all time.

Problem 7 We consider the (scaled) traffic model

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\Big(\rho(1-\rho)\Big) = 0$$

for modeling the traffic on a long single lane road. At position $x = 1$ there is a red traffic light behind which a queue starts to form. At time $t = 0$, when the traffic light turns green, the density of cars is given by

$$\rho_0(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 0 & \text{else.} \end{cases}$$

Sketch the characteristics of this equation, and show that a shock forms at $(x, t) = (1/2, 1/2)$. In addition, compute the solution of the equation for time $t < 1/2$ and show that the position $s(t)$ of the shock satisfies the differential equation

$$\dot{s} = \frac{s + t - 1}{2t}, \qquad s(1/2) = 1/2.$$