TMA4195 - MATHEMATICAL MODELING (FALL 2016).

DIESEL PARTICULATE FILTER

Introduction

In a diesel motor, the fuel is injected right before the power stroke. At this stage, due to compression, the air has reached a temperature sufficiently high for the ignition to start spontaneously. One of the drawback of this process is that the combustion is often incomplete and results in emissions of soot, which are components of the fuel that have not been burnt completely. These source of pollution is an important cause of diesel's harmful health effects. Diesel particulate filters have been developed to stop the emissions of soot particles into the atmosphere. The design of such filter is presented in Figure 1. Exhaust gas is sent from the motor into a honeycomb structure, which consist of aligned channels. There are two type of channels, the inlet and outlet channels, and an inlet channels is surrounded by outlet channels. All channels are closed at one end while the other end is open. The inlet channel is open towards the motor and the outlet channel is open towards the outside. The channels are separated by a porous material, usually ceramic, which is permeable to exhaust gas but not to the soot particles. The disposition of the channels forces all gas emissions to go through the porous walls and, in this way, the soot particle are stopped.

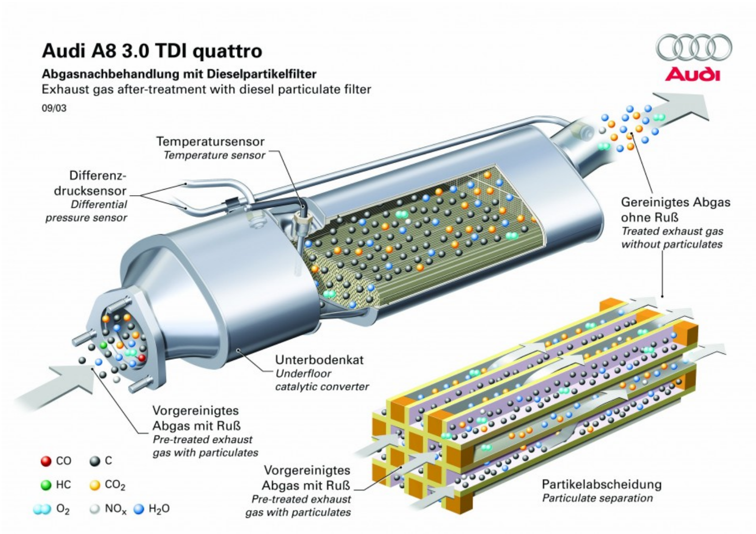

Figure 1. Diesel particulate filter

1. Filter efficiency

The efficiency of a filter can be measured as the ratio between the amount of particles that are filtered and the energy that is used to operate the filter. At

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the input and output of the filter, we can measure the velocity and the pressure of the gas. **Q1.1**: *From this data, can you evaluate the energy consumed by the filter?* **Q1.2**: *List up the parameters that plays a role in the operation of the filter. Proceed with a dimensional analysis and identify the dimensionless variables.* The number dimensionless variables gives the number of remaining parameters that can be optimized by a proper design of the filter.

2. Modeling equations for the gas in the channels

The governing equations for the gas moving in the channels are given by the conservation of mass,

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{2.1}$$

and the conservation of momentum,

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = f. \tag{2.2}$$

The term f consists of all the volumetric forces that are present in this system. **Q2.1**: *Derive* (2.1). **Q2.2**: *Derive* (2.2) *by starting from Newton's law of motion* $\frac{d}{dt}(mv) = f$ *for a single mass particle.* The external forces in f are the pressure forces and the viscous forces. We assume that the viscous forces acting on unit surface with orientation $\mathbf{n}$ are given by the following constitutive relation,

$$f_{\text{diss}} = \lambda(\nabla \cdot u)\mathbf{n} + 2\mu\varepsilon(u)\mathbf{n}, \tag{2.3}$$

where $\varepsilon(u)$, the symmetric gradient of the velocity field u, is defined as

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right).$$

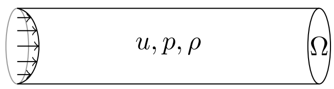
A fluid that that satisfies the constitutive relation (2.3) is called a Newtonian fluid. **Q2.3**: *Show that* f *is given by*

$$f = -\nabla p + \lambda\nabla(\nabla \cdot u) + 2\mu\nabla \cdot \varepsilon(u). \tag{2.4}$$

You may want to start by the one-dimensional case. At the boundary, we consider either no-flux boundary conditions or fixed-pressure conditions (in this case, gas may flow in and out from the domain). **Q2.4**: *Write down the boundary conditions in both cases.*

3. Approximation of the viscous force in the channel

We consider an infinite one-dimensional channel and denote by Ω the cross-section.



We assume that the flow is stationary and that u does not vary in the longitudinal direction. In this case, we expect a linear dependence of the viscous forces with respect to the average velocity, that is an expression of the form

$$\lambda\nabla(\nabla \cdot u) + 2\mu\nabla \cdot \varepsilon(u) = -a\bar{u},$$

where

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx \tag{3.1}$$

Q3.1: *Find an equation for* a. **Q3.2**: *Compute the coefficient* a *in the case where* Ω *is a disk.* **Q3.3**: *Compute the coefficient* a *in the case where* Ω *is a square, using the separation of variables.* **Q3.4**: *Compute numerically the coefficient* a *in the case where* Ω *is an hexagon.*

4. One dimensional approximation of the channel

We consider the one-dimensional approximation given by

$$\rho_t + (\rho u)_x = 0, \tag{4.1a}$$

$$(\rho u)_t + (\rho u^2)_x = -\frac{\partial p}{\partial x} - au. \tag{4.1b}$$

Q4.1: *How these equations relate to a full three-dimensional model? What do u now represent? What are the approximations that are made?*

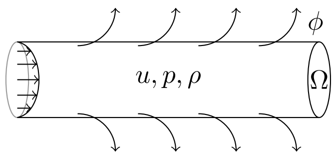
We model the gas as an ideal gas so that, at constant temperature, the pressure is proportional to the density,

$$p = r\rho T, \tag{4.2}$$

for some constant r. **Q4.2**: *Rewrite the equations* (4.1) *using this assumption* **Q4.3**: *Set up the stationary equations and try to solve them numerically* **Q4.4**: *Set up the equations for a small perturbation around the stationary state.*

5. Modeling mass loss from the wall

Now, we consider a constant mass loss from the wall.



Q5.1: *Incorporate this mass loss in the governing equation, for the full three-dimensional model and the reduced one-dimensional model.* **Q5.2**: *Solve numerically the steady-state equation in the 1D case.*

6. The porous media layer

We want to couple the flow from the channel to the porous media. In the porous media, we assume that we have a Darcy flow, that is, the flux depends linearly on the pressure gradient,

$$u = -\frac{K}{\mu}\nabla p, \tag{6.1}$$

where K is a symmetric matrix called the permeability tensor. **Q6.1**: *Derive the governing equation in the porous media, in terms of ρ, u and p.* We assume that the flow is traveling transversely from the channel in the porous media. We consider given pressures p_i and p_e on both sides of the porous media and assume that we have reached the steady-state.

Cylindrical channel Square channel

Let us denote by Γ, the surface at the interface between the channel and the porous media. We consider the incompressible case and define the total incoming flux as

$$U = \int_\Gamma u \cdot \mathbf{n}\, dx.$$

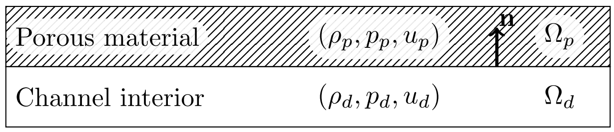
Q6.2: *Show that the flux U is proportional to the pressure drop $p_i - p_e$, that is, that there exists a constant κ such that*

$$U = \kappa(p_i - p_e). \tag{6.2}$$

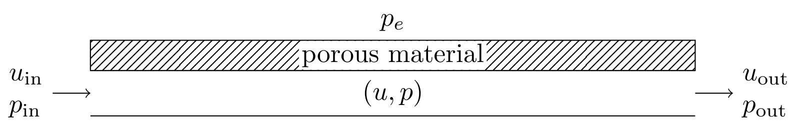
Q6.3: *Compute the constant κ in the case of a cylindrical channel.* **Q6.4**: *Compute the coefficient κ in the case of a square channel.* **Q6.5**: *Consider the case where we remove the incompressibility assumption.*

7. Coupling of the channel with a porous layer

The channel is surrounded by a porous layer through which the gas is going to flow.



Q7.1: *What are the interface conditions between the porous media and the channel?* We consider the stationary case and a given external pressure p_e. Let us use the assumptions and the results obtained in the previous section. Then, the interface condition simplifies to the continuity of pressure and we use the integrated model of the flow in the porous layer given by (6.2).



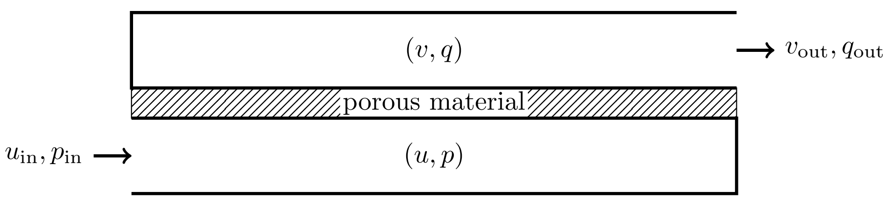
In addition, we assume that the gas is an ideal gas so that the pressure and the density are coupled through (4.2). Thus, the unknown are p and u. For the stationary one-dimensional model, the equations (4.1) then takes the form

$$A(p,u)\begin{pmatrix} p_x \\ u_x \end{pmatrix} = b(p,u), \tag{7.1}$$

where $A(p,u) \in \mathbb{R}^{2\times 2}$ and $b(p,u) \in \mathbb{R}^2$ depend non-linearly on p and u. **Q7.2**: *Find an expression for A and b.* **Q7.3**: *What kind of input/output can we consider?* **Q7.4**: *Set up a numerical method for the case where u_{in} and p_{out} are given.*

8. Coupling the inlet and outlet channel

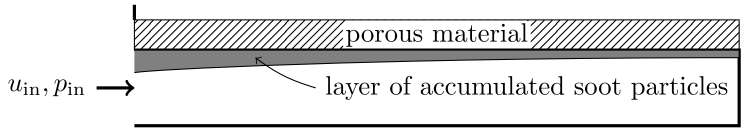
We couple an inlet and an outlet channel. We denote by v and q the flux and pressure in the outlet channel.



For simplicity, we can assume that the characteristics of the channels are the same. **Q8.1**: *Set up the equations for the reduced one-dimensional problem.* **Q8.2**: *Try to solve them numerically.*

9. Accumulation of particles

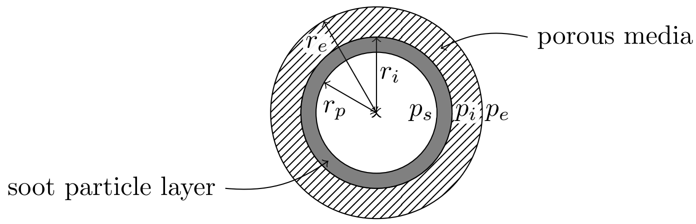
Let us denote by c the concentration of soot particles, which we define as the fraction of the mass of soot particle over the total mass of gas. Let us assume that the soot particles do not modify the property of the flow.



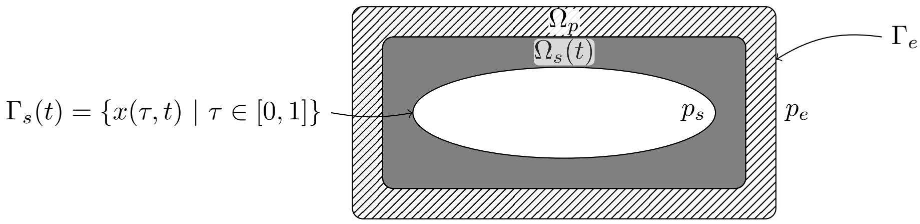
Q9.1: *What is the modeling equation that governs c, for the 3D model and the reduced one-dimensional model.* **Q9.2**: *Compute numerically the solution of the one-dimensional reduced model in the stationary case.* **Q9.3**: *Compute the amount of accumulated particles as a function of t and x*

10. Filter clogging

The soot particles accumulate on the porous wall. The cross-section of the channel is therefore reduced. Moreover, the layer of soot particles affects the permeability of the porous wall so that the coefficient κ derived in (6.2) is reduced.



Let us assume that the permeability of the region of accumulated soot particles can be determined. **Q10.1**: *Using the same assumptions as in Section 6 when studying the porous media layer, compute κ as a function of r_p.* We consider again the one-dimensional reduced model. At a given x, the accumulated mass of soot particles is directly related the radius r_p. **Q10.2**: *Couple the flow equations with the particle transport equation through κ.* **Q10.3**: *Solve numerically the equations that you obtain and compute the loss of efficiency of the filter a function of time.* We consider now a more general geometry for the channel. We denote the evolving domain which contains the accumulated particles by $\Omega_s(t)$ and its interface with the interior of the channel by $\Gamma_s(t)$. We represent the interface $\Gamma_s(t)$ as a parametric curve defined through the function $x(\tau, t)$ for the parameter $\tau \in [0, 1]$.



Let us assume that the gas is incompressible. **Q10.4**: *Derive modeling equations for the pressure in the porous media and the accumulated soot region and for the curve $x(t, s)$.* **Q10.5**: *Design and implement a numerical scheme to solve these equations.*