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Solutions to exercise set 1

1 a) Using the substitution in the hint, we see that

$$\tilde{\pi}_1 = \frac{\pi_2^2}{\pi_1}, \qquad \tilde{\pi}_2 = \sqrt[3]{\pi_1\pi_2},$$
$$\pi_1 = \frac{\tilde{\pi}_2^2}{\sqrt[3]{\tilde{\pi}_1}}, \qquad \pi_2 = \tilde{\pi}_2\sqrt[3]{\tilde{\pi}_1}.$$

We also note that

$$\frac{cd}{ab} = \frac{\pi_2}{\pi_1}, \qquad \frac{ab}{cd} = \frac{\tilde{\pi}_2}{\sqrt[3]{\tilde{\pi}_1}^2},$$

thus we can write

$$\varphi(\pi_1,\pi_2) = \frac{cd}{ab}\phi(\tilde{\pi}_1,\tilde{\pi}_2) = \frac{\pi_2}{\pi_1}\phi\Big(\frac{\pi_2^2}{\pi_1}, \sqrt[3]{\pi_1\pi_2}\Big),$$
$$\phi(\tilde{\pi}_1,\tilde{\pi}_2) = \frac{ab}{cd}\varphi(\pi_1,\pi_2) = \frac{\tilde{\pi}_2}{\sqrt[3]{\tilde{\pi}_1}^2}\varphi\Big(\frac{\tilde{\pi}_2^2}{\sqrt[3]{\tilde{\pi}_1}}, \tilde{\pi}_2\sqrt[3]{\tilde{\pi}_1}\Big).$$

b) The three dimensionless combinations of ψ are not independent:

$$\Big(abe\Big)^3 = \Big(\frac{ce^3}{a^2d}\Big)\Big(\frac{a^5b^3d}{c}\Big).$$

2 The rank of the dimension matrix is 3 and hence we can use as core variable any 3 R_i whose columns are independent.

Note that

$$2\overbrace{\begin{pmatrix}1\\-1\\1\end{pmatrix}}^{R_2} = \overbrace{\begin{pmatrix}2\\-2\\2\end{pmatrix}}^{R_6}$$

so we remove R_6 for the time being. Then we note that only R_2 and R_4 contains dimension F_3 and hence one of these must be present in any choice of core variables. Let us then try

R_2 :	$R_2R_1R_3$	~~$R_2R_3R_4$~~	$R_2R_4R_5$
	$R_2R_1R_4$	$R_2R_3R_5$	
	$R_2R_1R_5$		
$R_4(-R_2)$:	$R_3R_1R_4$	$R_3R_4R_5$	
	$R_4R_1R_5$		

It is easy to see that all these combinations have independent columns in the dimension matrix, except $R_2R_3R_4$.

Taking into account R_6, we find the combinations as for R_2, but with R_6 replacing R_2. In all 13 possible choices of core variables.

3 Using the information provided in the problem, we assume

$$F = f\left(U, L, W, D, \rho, \nu, g\right).$$

The dimension matrix follows immediately and is shown in Table 1.

	F	U	L	W	D	ρ	ν	g
m	1	1	1	1	1	-3	2	1
s	-2	-1	0	0	0	0	-1	-2
kg	1	0	0	0	0	1	0	0

Table 1: Dimension matrix

The rank is 3, and there are several possibilities for core variables (avoiding F): (U, L, ρ), (g, D, ρ), (ν, ρ, W), ... However, if one aims for the Froude and Reynolds numbers, the choice (U, L, ρ) looks reasonable. With 8 variables, there are $8 - 3 = 5$ dimensionless combinations.

Since Re involves ν and Fr involves g, it is easy to arrive at the formula

$$F = \rho U^2 L^2 \times \Phi\left(\mathrm{Re}, \mathrm{Fr}, \frac{W}{L}, \frac{D}{L}\right).$$

The scale model keeps W/L and D/L unchanged, so if we forget those, we need to map the function

$$\pi_1 = \frac{F}{\rho U^2 L^2} = \Phi\left(\mathrm{Re}, \mathrm{Fr}\right)$$

for the range of Re and Fr typical for the original ship. Assume that length of the model, L_m, is equal to rL, where r is about 10^{-2}. If we aim to keep Fr, we have to run the model with $U_m = \sqrt{r}U$, which looks feasible. However, for the same ν, the Reynolds number would then be a factor $r^{3/2}$ off. The only way to compensate this would actually be find a fluid with a correspondingly small viscosity, but this does not exist. If we start by keeping the Reynold number (rather unrealistic!) we run into similar problems (Read more about ship resistance on the Internet).

4 Let us begin setting up the dimension matrix for the physical quantities involved in the problem.

	ω	l	ρ	F
kg	0	0	1	1
m	0	1	-1	1
s	-1	0	0	-2

This matrix has rank 3. We easily find three linearly independent columns, for example 1, 2 and 3 and so we choose ω, l and ρ as core variables. The first dimensionless combination we find is $\pi_1 = F/(\rho^x l^y \omega^z)$. The unknowns can be found easily and are $x = 1$, $y = 2$ and $z = 2$, that is $\pi_1 = F/(\rho l^2 \omega^2)$. If there is relationship between these quantities, it has to be of the form $f(\pi_1) = 0$: that is, π_1 is a constant. This implies

$$\omega = C\sqrt{\frac{F}{\rho l^2}}.$$

We had been given that $F \propto l - l_0$ (except the deformation at rupture). In the area where the observed eigenfrequencies are close to a constant, we can set

$$F \approx F_0 \frac{l - l_0}{l_0}$$

for an fixed constant F_0. Since the total mass has to be constant and independent of the length, we get $\rho l = \rho_0 l_0$. Let us plug this into the expression for ω to get

$$\omega \approx C\sqrt{\frac{F_0(l - l_0)}{l^2 \rho_0 l_0^2 / l}} = C\sqrt{\frac{F_0}{\rho_0 l_0^2}}\sqrt{1 - \frac{l_0}{l}}.$$

Whenever $l \gg l_0$ we have

$$\omega \approx C\sqrt{\frac{F_0}{\rho_0 l_0^2}}\left(1 - \frac{l_0}{2l}\right),$$

and this means that the frequency will be close to a constant in this situation.

When we get closer to the deformation at rupture, we know that the force F increases faster than $l - l_0$, and thus it is reasonable to expect a certain growth for the frequency.

From Calculus 4 we remember the partial differential equation (called the wave equation and used to describe the phenomenon we are actually dealing with)

$$\frac{\partial^2 u}{\partial t^2} = \frac{F}{\rho}\frac{\partial^2 u}{\partial x^2},$$

with basic solution

$$u\left(x, t\right) = \sin\left(\pi \frac{x}{l}\right)\sin(\omega t).$$

If we plug this solution in, we will find out that C has to be equal to π.