Norwegian University of Science and Technology
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TMA4195 Mathematical Modelling Autumn 2017

Exercise set 2
Exercise session 2017-9-7

1 Let

$$u(x) = \mathrm{e}^{-10x} + \mathrm{e}^{-100x} \quad \text{for} \quad x \in [0, 1]. \tag{1}$$

Suggest (natural) scales for x and indicate where in $[0, 1]$ their use is reasonable.

2 (Problem 7 p. 32 in Logan)
A rocket blasts off from the earth's surface. During the initial phase of flight, fuel is burned at the maximum possible rate α, and the exhaust gas is expelled downward with velocity β relative to the velocity of the rocket. The motion is governed by the following set of equations:

$$m'(t) = -\alpha, \quad m(0) = M, \tag{2}$$

$$v'(t) = \frac{\alpha\beta}{m(t)} - \frac{g}{\left(1 + \frac{x(t)}{R}\right)^2}, \quad v(0) = 0, \tag{3}$$

$$x'(t) = v(t), \quad x(0) = 0 \tag{4}$$

where $m(t)$ is the mass of the rocket, $v(t)$ is the upward velocity, $x(t)$ is the height above the earth's surface, M is the initial mass, g is the gravitational constant, and R is the radius of the earth. Reformulate the problem in terms of dimensionless variables using appropriate scales for m, x, v, t.
(*Hint*: Scale m and x by obvious choices; then choose the time scale and velocity scale to ensure that the terms in the v equation are of the same order as well as the terms in the x equation. Assume that the acceleration is due primarily to fuel burning and that the gravitational force is small in comparison.)

3 Let the line of real numbers represent an infinitely long river. Suppose sewage is poured out uniformly over the distance $(0, L)$ at the instant $t^* = 0$. If we let $u^*(x^*, t^*)$ represent the concentration of sewage at position x^* and time t^*, we can model the transport of the pollution by the PDE:

$$u^*_{t^*} + cu^*_{x^*} = \kappa u^*_{x^*x^*}, \qquad u^*(x^*, 0) = \begin{cases} U, & \text{if } 0 < x^* < L, \\ 0, & \text{else,} \end{cases}$$

for two non-zero constants c and κ. It is known (by the maximum principle) that $0 \leq u^* \leq U$ for $t^* > 0$. A natural scaling in this problem is $x^* = Lx$.

For the two following cases, determine the scales for u^* and t^* and find the scaled equation:

(a) $|\kappa| \ll |cL|$.

(b) $|\kappa| \gg |cL|$.

4 (Problem 4.2.5 p. 55 in Krogstad)
Case B in the discussion in Krogstad of the falling sphere in a fluid (section 2.3.2) led to the equation

$$2\ddot{x} + \epsilon\dot{x} = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = 0, \qquad 0 < \epsilon \ll 1.$$

This equation has the exact solution

$$x_{sol}(t) = \frac{2}{\epsilon^2}\left(e^{-\frac{1}{2}\epsilon t} - 1\right) + \frac{t}{\epsilon}.$$

(a) Determine x_0, x_1 and x_2 in the regular perturbation expansion

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots,$$

and show that it agrees with the start of the power series development in ϵ of the exact solution.

(b) An approximate solution $x_a(t, \epsilon)$ is a uniform approximation to the exact solution, x_{sol}, on the interval $[0, 1]$ if

$$\lim_{\epsilon \to 0}\left(\max_{t \in [0,1]} |x_a(t) - x_{sol}(t)|\right) = 0.$$

Does this apply to $x_a(t, \epsilon) = x_0(t) + \epsilon x_1(t)$? What if we replace $[0, 1]$ with $[0, \infty)$?

5 (Problem 4.2.7 p. 54 in Krogstad)
This problem is somewhat similar to the sphere falling in a fluid (the scaling model problem without gravity), but in this case the friction is nonlinear. The equation reads

$$m\frac{dv^*}{dt^*} = -av^* + bv^{*2}, \tag{5}$$

and

$$v^*(0) = V_0. \tag{6}$$

We are told that $a, b > 0$, and also that $bV_0 \ll a$.

(a) First find the (obvious) scale for v^* and then the scale for time, T, from the simplified equation $m\frac{dv^*}{dt^*} = -av^*$ and the "rule of thumb"

$$T = \frac{\max|v^*|}{\max|dv^*/dt^*|}.$$

Show that this scaling leads to the equation

$$\frac{dv}{dt} = -v + \varepsilon v^2,\ v(0) = 1,\ \varepsilon \ll 1. \tag{7}$$

(b) Determine v_0 and v_1 of the series expansion $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$. Is this result reasonable for all $t > 0$ when the general solution of $\dot{y} = -y + \varepsilon y^2 = 0$ is

$$y(t) = \frac{e^{-t}}{C + \varepsilon e^{-t}},$$

and C is a constant?