TMA4195
Mathematical Modelling
Autumn 2017

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Solutions to exercise set 2

1 In our notation we write

$$u(x^*) = \mathrm{e}^{-10x^*} + \mathrm{e}^{-100x^*}, \quad x^* \in [0,1]$$

Natural scalings:

$$\begin{aligned}
&1.) \quad x^* = \frac{1}{100}x_1 \quad (\mathrm{e}^{-100x^*} \sim 1, \text{ when } x_1 \sim 1)\\
&2.) \quad x^* = \frac{1}{10}x_2 \quad (\mathrm{e}^{-10x^*} \sim 1, \text{ when } x_2 \sim 1)\\
&3.) \quad x^* = 1 \cdot x_3 \quad (u \sim 0, \text{ when } x_3 \sim 1)
\end{aligned}$$

Regions

$$\begin{aligned}
1.) \quad & x_1 \in [0,2] \Rightarrow x^* \in \left[0, \frac{2}{100}\right]\\
& u(x_1) = \mathrm{e}^{-\frac{1}{10}x_1} + \mathrm{e}^{-x_1} \approx 1 + \mathrm{e}^{-x_1}\\
& u(x^*) \approx 1 + \mathrm{e}^{-100x^*}\\
2.) \quad & x_2 \in \left[\frac{2}{10}, 2\right] \Rightarrow x^* \in \left[\frac{2}{100}, \frac{2}{10}\right]\\
& u(x_2) = \mathrm{e}^{-x_2} + \mathrm{e}^{-10x_2} \approx \mathrm{e}^{-x_2}\\
& u(x^*) \approx \mathrm{e}^{-10x^*}\\
3.) \quad & x_3 = x^* \in \left[\frac{2}{10}, 1\right]\\
& u(x_3) = \mathrm{e}^{-10x_3} + \mathrm{e}^{-100x_3} \approx 0
\end{aligned}$$

2 In our notation the problem can be written as

(1) $$m^{*\prime}(t^*) = -\alpha, \quad m^*(0) = M,$$

(2) $$v^{*\prime}(t^*) = \frac{\alpha\beta}{m^*(t^*)} - \frac{g}{\left(1 + \frac{x^*(t^*)}{R}\right)^2}, \quad v^*(0) = 0,$$

(3) $$x^{*\prime}(t^*) = v^*(t^*), \quad x^*(0) = 0$$

the natural scalings for x^* and m^* are

$$x^* = Rx$$
$$m^* = Mm$$

Assuming acceleration is mainly due to the rocket engine, we neglect for the time being the gravity term. $[v^*]$ and $[t^*]$ are then chosen so that the remaining terms in (2) balances and the terms in (3) balances:

$$(2) \rightsquigarrow [v^{*\prime}] = \left[\frac{\alpha\beta}{m^*}\right] \rightsquigarrow \frac{[v^*]}{[t^*]} = \frac{\alpha\beta}{M} \tag{4}$$

$$(3) \rightsquigarrow [x^{*\prime}] = [v^*] \rightsquigarrow \frac{R}{[t^*]} = [v^*] \tag{5}$$

Solving, we find that

$$[t^*] = \sqrt{\frac{RM}{\alpha\beta}} \quad \text{and} \quad [v^*] = \sqrt{\frac{R\alpha\beta}{M}}$$

Using

$$x^* = Rx, \quad m^* = Mm, \quad t^* = \sqrt{\frac{RM}{\alpha\beta}}t, \quad v^* = \sqrt{\frac{R\alpha\beta}{M}}v$$

we find the scaled equations

$$\begin{aligned} m'(t) &= -\sqrt{\frac{R\alpha}{M\beta}}, \quad m(0) = 1, \\ v'(t) &= \frac{1}{m(t)} - \frac{Mg}{\alpha\beta}\frac{1}{(1+x)^2}, \quad v(0) = 0, \\ x'(t) &= v(t), \quad x(0) = 0 \end{aligned}$$

3 **(a)** By the maximum principle it is natural to set $u^* = Uu$. Setting $x^* = Lx$ and $t^* = Tt$, and substituting the scaled variables into the original equation we obtain

$$\frac{1}{T}u_t + \frac{c}{L}u_x = \frac{\kappa}{L^2}u_{xx},$$

where we have divided by U. By the *scaling assumption* $u_t, u_x, u_{xx} \sim 1$, and by the assumption $\kappa/L^2 \ll c/L$, we conclude that the third term is dominated by the second. By the differential equation, the first and second term must be of the same order; it is then natural to balance their respective coefficients by setting

$$T = \frac{L}{c}.$$

Multiplying with L/c we obtain the scaled equation

$$u_t + u_x = \varepsilon u_{xx},$$

where $\varepsilon = \frac{\kappa}{cL} \ll 1$.

(b) Again, by the maximum principle it is natural to set $u^* = Uu$. Setting $x^* = Lx$ and $t^* = Tt$, and substituting the scaled variables into the original equation we yet again obtain

$$\frac{1}{T}u_t + \frac{c}{L}u_x = \frac{\kappa}{L^2}u_{xx},$$

where we have divided by U. By the *scaling assumption* $u_t, u_x, u_{xx} \sim 1$, and by the assumption $\kappa/L^2 \gg c/L$, we conclude that the second term is dominated by the third. By the differential equation, the first and third term must be of the same order; it is then natural to balance their respective coefficients by setting

$$T = \frac{L^2}{\kappa}.$$

Multiplying with L^2/κ we obtain the scaled equation

$$u_t + \varepsilon u_x = u_{xx},$$

where $\varepsilon = \frac{cL}{\kappa} \ll 1$.

4 **(a)** Let $x(t) = \sum_{n=0}^{\infty} \epsilon^n x_n(t)$, and insert this into the equation. Differentiating and reordering gives

$$2\ddot{x}_0 - 1 + \sum_{n=1}^{\infty} \epsilon^n \left(2\ddot{x}_n + \dot{x}_{n-1}\right) = 0.$$

This should hold for all $0 < \epsilon \leq 1$ so the expression for each power of ϵ has to be zero. We get a system of ODE's

$$\begin{aligned} 2\ddot{x}_0 - 1 &= 0, \\ 2\ddot{x}_n + \dot{x}_{n-1} &= 0, \qquad n = 1, 2, 3, \dots . \end{aligned}$$

The initial condition should hold for all $0 < \epsilon \leq 1$, and thus $x_i(0) = 0, \dot{x}_i(0) = 0$. We integrate the system of ODE's to get

$$\begin{aligned} x_0(t) &= \frac{1}{4} t^2, \\ x_n(t) &= -\left(-\frac{1}{2}\right)^{n+1} \frac{t^{n+2}}{(n+2)!}, \qquad n = 1, 2, 3, \dots . \end{aligned}$$

The power series expansion of the exact solution is $x_{sol}(t) = \frac{2}{\epsilon^2} \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n \frac{(\epsilon t)^n}{n!}$. If we compute the first few terms in the power series and compare them with $x_0 + \epsilon x_1 + \epsilon^2 x_2$, we see that they coincide.

(b) Let $t \in [0, 1]$, then

$$|x_{sol} - x_a| = |\frac{2}{\epsilon^2} \sum_{n=4}^{\infty} \left(-\frac{1}{2}\right)^n \frac{(\epsilon t)^n}{n!}| \leq \frac{\epsilon^2}{192},$$

and x_a is a uniform approximation to x_{sol} *on* $[0, 1]$. The key step is that we can bound t^n, if not we could choose a big t and make the "error" arbitrarily large. If we consider the set $[0, \infty)$ instead of $[0, 1]$ we run into problems as $\sup_{t \in [0,\infty)} |x_{sol}(t) - x_a(t)| = \infty$ for all $0 < \epsilon \leq 1$. Here we have to use supremum instead of maximum as the function never attains its maximum. Supremum is in some sense "maximum including limits" (the least upper bound).

5 (a) From the problem's nature we have $0 \leq v^*(t) \leq V_0$. Then, V_0 will be a scale for v^*, and moreover

$$\frac{|bv^{*2}|}{|av^*|} \leq \frac{bV_0}{a} \ll 1.$$

We find a time scale from the simplified equation $m\frac{dv^*}{dt^*} + av^* = 0$ with solution $v^*(t^*) = A\exp\left(-\frac{a}{m}t^*\right)$, that is $T = \frac{m}{a}$. Alternatively, and this is easier, we find this scale by balancing the first and second term in equation (5) in the problem set (set $v^* = Vv$ and $t^* = Tt$):

$$m\frac{dv^*}{dt^*} \sim av^* \quad \Rightarrow \quad m\frac{V}{T}\frac{dv}{dt} \sim aVv \quad \underset{v,\dot{v}\sim 1}{\rightsquigarrow} \quad m\frac{V}{T} \sim aV \quad \Rightarrow \quad T \sim \frac{m}{a}.$$

Using this scaling, we obtain the equation in the desired form and $\varepsilon = bV_0/a \ll 1$.

(b) Plugging in $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$ into the equation, we get that

$$\begin{aligned} O\left(\varepsilon^0\right): &\quad \dot{v}_0 = -v_0, \\ O\left(\varepsilon^1\right): &\quad \dot{v}_1 = -v_1 + v_0^2. \end{aligned}$$

Considering the initial condition,

$$\begin{aligned} v_0(t) &= \mathrm{e}^{-t}, \\ v_1(t) &= \mathrm{e}^{-t} - \mathrm{e}^{-2t}, \end{aligned}$$

or

$$v(t) = \mathrm{e}^{-t} + \varepsilon\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right) + O\left(\varepsilon^2\right).$$

This is the so-called regular perturbation. We have shown through some examples that the approximated solution not always is reasonable when $t \to \infty$, and we need to check the validity of the approximated solution.

From the theory we know that the exact solution has the form

$$v_{\text{ex}}(t) = \frac{\mathrm{e}^{-t}}{1 - \varepsilon\left(1 - \mathrm{e}^{-t}\right)},$$

and since $0 \leq 1 - \mathrm{e}^{-t} < 1$ for $t \geq 0$, we can write the solution as a convergent geometric series.

$$v_{\text{ex}}(t) = \mathrm{e}^{-t}\sum_{k=0}^{\infty}\left(\varepsilon\left(1 - \mathrm{e}^{-t}\right)\right)^k$$

The initial terms in the perturbation expansion coincide with the initial terms in the series above, and we have:

$$v_{\text{ex}}(t) - \left(v_0(t) + \varepsilon v_1(t)\right) = \mathrm{e}^{-t}\sum_{k=2}^{\infty}\left(\varepsilon\left(1 - \mathrm{e}^{-t}\right)\right)^k \leq \mathrm{e}^{-t}\varepsilon^2\sum_{m=0}^{\infty}\varepsilon^m = \frac{\varepsilon^2\mathrm{e}^{-t}}{1-\varepsilon} \leq \frac{\varepsilon^2}{1-\varepsilon}.$$

Thus, we have

$$\lim_{\epsilon\to 0}\left(\sup_{t>0}\left|v_{\text{ex}}(t) - \left(v_0(t) + \varepsilon v_1(t)\right)\right|\right) = 0,$$

and so $v_a(t) = v_0(t) + \varepsilon v_1(t)$ is a uniform approximation to the exact solution on the domain $t > 0$.