TMA4195 Mathematical Modelling Autumn 2017

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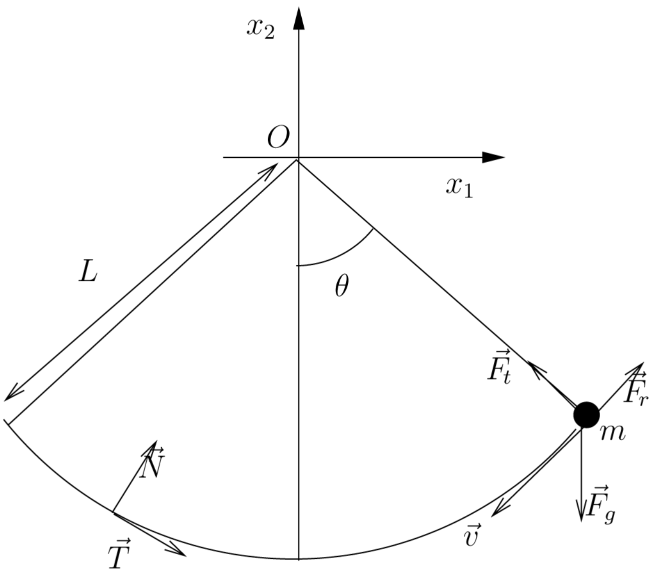
Exercise set 3
Exercise session 2017-9-14

1 The 2D pendulum



A pendulum of mass m and length L and point of fixation O is set in motion. The position $\vec{x}$ of the pendulum is constrained to be on the circle

$$S = \{\vec{x} : \quad |\vec{x}| = L\}$$

Let $\vec{T}(\vec{x})$ be the unit tangent of S at $\vec{x}$ in the counterclockwise sense, and $\vec{N}(\vec{x}) = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{L}$ be the outward pointing unit normal of S at $\vec{x}$. Since $|\vec{x}|^2 = L^2$, $0 = \frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}|^2 = 2\vec{x} \cdot \dot{\vec{x}} = 2L\vec{N} \cdot \dot{\vec{x}}$ and hence $\vec{N} \cdot \dot{\vec{x}} = 0$ and $\vec{T} = \pm\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$.

The forces acting on the pendulum are the following:

1. Gravity: $\vec{F}_g = -mg\vec{e}_2$
2. Air resistance: $\vec{F}_r = -k\dot{\vec{x}}$
3. String tension: $\vec{F}_t = c(\vec{x})\vec{N}(\vec{x})$

where $\vec{e}_2$ is an unit vector. Note that $\vec{N}$ is the tangent of the string, and that

$$\vec{F}_r \cdot \vec{N} = -k\vec{N} \cdot \dot{\vec{x}} = 0$$

a) Use Newton's 2. law in $\vec{N}$ and $\vec{T}$ directions to find the equation of motion

$$mL\ddot{\theta} = -mg\sin\theta - kL\dot{\theta} \tag{1}$$

where $\theta = \phi - \frac{3\pi}{2}$ and ϕ is the polar angle. $\theta = 0 \Rightarrow \vec{x} = -L\vec{e}_2$.
Hint: In polar coordinates

$$\vec{x}(t) = L \begin{pmatrix} \cos\phi(t) \\ \sin\phi(t) \end{pmatrix},$$

$$\vec{T}(\vec{x}(t)) = \begin{pmatrix} -\sin\phi(t) \\ \cos\phi(t) \end{pmatrix}$$

The pendulum is set in motion from an initial angle $\theta(0) = \alpha$, the initial velocity is 0.

b) Scale the initial value problem for (1) when friction is small compared to gravity, and $|\theta(0)| = |\alpha| \ll 1$.
(Hint: Recall that $\sin(x) \approx x$, for small x.)

Let $k = 0$ and $\alpha = \epsilon$ be small, then a good scaling of the initial value problem for (1) is

$$\ddot{\theta} = -\frac{1}{\epsilon}\sin(\epsilon\theta), \quad \theta(0) = 1,\ \dot{\theta}(0) = 0. \tag{2}$$

Assume

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \dots$$

c) Write down the initial value problems for θ_0, θ_1, and θ_2. Solve for θ_0, θ_1, and verify that

$$6\theta_2 = \frac{1}{32}(\cos t - \cos 3t) + \frac{3}{8}t\sin t.$$

Hint: $\cos^3 t = \frac{1}{4}(3\cos t + \cos 3t)$.

The perturbation solution for (2) is

$$\theta = \cos t - \epsilon^2\frac{1}{6}\left(\frac{\cos 3t - \cos t}{32} - \frac{3}{8}t\sin t\right) + \dots.$$

In this problem, the exact solution θ is bounded, but the perturbation solution contains an unbounded term

$$\epsilon^2\frac{1}{6}\cdot\frac{3}{8}\cdot t\sin t.$$

Such terms are called <u>secular</u> terms and will destroy the approximation when t is big. The approximation is only valid for t such that $\epsilon^2\frac{3}{48}t\sin t \sim \epsilon^2\frac{t}{10} \ll 1$.

To avoid secular terms in the perturbation expansion, one can use the Poincare-Lindstedt method: Assume

$$\theta(t) = \theta_0(\omega t) + \epsilon\theta_1(\omega t) + \epsilon^2\theta_2(\omega t) + \dots$$

and

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots,$$

and choose $\omega_1, \omega_2, \dots$ in order to cancel the secular terms.

d) Find equations for $\theta_0, \theta_1, \ldots$ and show that

$$\theta_0(t) = \cos t.$$

Determine θ_1 and θ_2 by choosing $\omega_1, \omega_2, \ldots$ to avoid unbounded terms.

2 Consider the problem

$$\ddot{x} + 2\varepsilon\dot{x} + \varepsilon x = 0, \qquad x(0) = 0, \qquad \dot{x}(0) = 1 - \varepsilon.$$

Determine x_0 and x_1 in the regular perturbation expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots.$$

3 (Problem 2, p. 23 in A. W. Bush: *Perturbation Methods for Engineers and Scientists*)
Obtain a two-term perturbation expansion for the solution of

$$\dot{y} - y - \varepsilon y^2 \mathrm{e}^{-t} = 0, \qquad y(0) = 1.$$

4 (Exercise 5 p. 299 in Lin & Segel)
Use singular perturbation theory to obtain outer, inner, and composite expansions to the solution of the problem

$$\epsilon u'' - (2 - x^2)u = -1, \qquad u(-1) = u(1) = 0.$$

REMARK. It is sufficient to solve the differential equation on $(0, 1)$ subject to the boundary conditions $u'(0) = 0, u(1) = 0$. Why?