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Solutions to exercise set 3

1 a) Note that there is no motion in the normal direction, and hence the normal forces are at equilibrium:

$$0 = m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}\cdot\vec{N} = \vec{F}_g\cdot\vec{N} + \vec{F}_r\cdot\vec{N} + \vec{F}_t\cdot\vec{N}$$

Since $\vec{F}_t$ is a normal force, it is determined by this equation: $\vec{F}_t = -(\vec{F}_g\cdot\vec{N})\vec{N}$, $(\vec{F}_r\cdot\vec{N} = 0)$ we will not need this equation here. For the tangential components, Newton's 2nd law yields

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}\cdot\vec{T} = \vec{F}_g\cdot\vec{T} + \vec{F}_r\cdot\vec{T} + \vec{F}_t\cdot\vec{T} \tag{1}$$

Note that

$$\begin{aligned}\vec{F}_g\cdot\vec{T} &= -mg\vec{e}_2\cdot\vec{T} = -mg\cos\phi = -mg\sin\theta\\ \vec{F}_t\cdot\vec{T} &= 0\end{aligned}$$

In polar coordinates we find that

$$\begin{aligned}\vec{x}(t) &= L\begin{pmatrix}\cos\phi(t)\\ \sin\phi(t)\end{pmatrix}\\ \dot{\vec{x}}(t) &= L\begin{pmatrix}-\sin\phi(t)\\ \cos\phi(t)\end{pmatrix}\dot{\phi}(t)\\ \ddot{\vec{x}}(t) &= L\begin{pmatrix}-\cos\phi(t)\\ -\sin\phi(t)\end{pmatrix}\dot{\phi}(t)^2 + L\begin{pmatrix}-\sin\phi(t)\\ \cos\phi(t)\end{pmatrix}\ddot{\phi}(t)\\ \vec{T}(\vec{x}(t)) &= \begin{pmatrix}-\sin\phi(t)\\ \cos\phi(t)\end{pmatrix}\\ \ddot{\vec{x}}\cdot\vec{T} &= 0 + L\cdot 1\cdot\ddot{\phi}(t)\\ \vec{F}_r\cdot\vec{T} &= -kL\dot{\phi}(t)\end{aligned}$$

Since $\theta = \phi - \frac{3\pi}{2}$ we find that (1) is equivalent to

$$mL\ddot{\theta} = -mg\sin\theta - kL\dot{\theta}$$

which is what we should show.

b) Since $\max|\theta| = \alpha$, we choose

$$\theta = \alpha\bar{\theta}.$$

The time scale

$$t = T\bar{t},$$

is determined from balancing acceleration and gravity terms (friction is small) in the scaled equation:

$$mL\frac{\alpha}{T^2}\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}\bar{t}^2} = -mg\sin(\alpha\bar{\theta}) - kL\frac{\alpha}{T}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{t}}, \tag{2}$$

i.e.

$$mL\frac{\alpha}{T^2}\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}\bar{t}^2} \sim -mg\sin(\alpha\bar{\theta}) \underset{\ddot{\bar{\theta}}\sim 1}{\overset{\sin\alpha\bar{\theta}\approx\alpha\bar{\theta}\sim\alpha}{\Longrightarrow}} mL\frac{\alpha}{T^2} \sim mg\alpha \Rightarrow T \sim \sqrt{\frac{L}{g}}.$$

We set $\theta = \alpha\bar{\theta}$ and $t = \sqrt{\frac{L}{g}}\bar{t}$ and equation (2) becomes

$$\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}\bar{t}^2} = -\frac{1}{\alpha}\sin(\alpha\bar{\theta}) - \frac{k}{m}\sqrt{\frac{L}{g}}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{t}} \tag{3}$$

For the initial conditions we have

$$\alpha\bar{\theta}(0) = \alpha \Rightarrow \bar{\theta}(0) = 1 \quad \text{and} \quad \dot{\bar{\theta}}(0) = 0.$$

c) We are given

$$\ddot{\theta} = -\frac{1}{\epsilon}\sin(\epsilon\theta), \quad \theta(0) = 1,\ \dot{\theta}(0) = 0. \tag{4}$$

Insert $\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots$ into (4) and expand:

$$\begin{aligned}
&\ddot{\theta}_0 + \epsilon\ddot{\theta}_1 + \epsilon^2\ddot{\theta}_2 = -\frac{1}{\epsilon}\sin(\epsilon(\theta_0 + \epsilon\theta_1 + \cdots))\\
&= -\frac{1}{\epsilon}\left(\epsilon(\theta_0 + \epsilon\theta_1 + \cdots) - \frac{1}{6}\epsilon^3(\theta_0 + \epsilon\theta_1 + \cdots)^3 + \cdots\right)\\
&= -\theta_0 - \epsilon\theta_1 - \epsilon^2(\theta_2 - \frac{1}{6}\theta_0^3) - \epsilon^3(\theta_3 - \frac{1}{2}\theta_1\theta_0^2) + \cdots\\
&\theta_0(0) + \epsilon\theta_1(0) + \cdots = 1\\
&\dot{\theta}_0(0) + \epsilon\dot{\theta}_1(0) + \cdots = 0
\end{aligned}$$

Equate terms of equal order in ϵ :

$$\begin{aligned}
\mathcal{O}(1):&\quad \ddot{\theta}_0 = -\theta_0; \quad \theta_0(0) = 1, \quad \dot{\theta}_0(0) = 0\\
\mathcal{O}(\epsilon):&\quad \ddot{\theta}_1 = -\theta_1; \quad \theta_1(0) = 0, \quad \dot{\theta}_1(0) = 0\\
\mathcal{O}(\epsilon^2):&\quad \ddot{\theta}_2 = -\theta_2 + \frac{1}{6}\theta_0^3; \quad \theta_2(0) = 0, \quad \dot{\theta}_2(0) = 0\\
\mathcal{O}(\epsilon^3):&\quad \ddot{\theta}_3 = -\theta_3 + \frac{1}{2}\theta_1\theta_0^2; \quad \theta_3(0) = 0, \quad \dot{\theta}_3(0) = 0
\end{aligned}$$

The solutions are

$$\theta_0 = \cos t,\ \theta_1 = 0$$

For θ_2 we find that

$$\ddot{\theta}_2 = -\theta_2 + \frac{1}{6}\cos^3 t \overset{\text{Hint}}{=} -\theta_2 + \frac{1}{24}(3\cos t + \cos 3t)$$

We differentiate the solution given in the text twice

$$\ddot{\theta}_2 = -\frac{1}{192}(\cos t - 9\cos 3t) + \frac{1}{16}(2\cos t - t\sin t)$$

and check that it satisfies the previous equation with $\theta_2(0) = 0 = \dot{\theta}_2(0)$.

d) Inserting into (4) we find

$$\omega^2(\ddot{\theta}_0 + \epsilon\ddot{\theta}_1 + \cdots) = \cdots = -\frac{1}{\epsilon}\sin(\epsilon(\theta_0 + \epsilon\theta_1 + \cdots))$$

or

$$(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2(\ddot{\theta}_0 + \epsilon\ddot{\theta}_1 + \epsilon^2\ddot{\theta}_2 + \cdots)$$
$$= -(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots) + \frac{1}{6}\epsilon^2(\theta_0 + \epsilon\theta_1 + \cdots)^3 + \cdots$$

or

$$\ddot{\theta}_0 + \epsilon(2\omega_1\ddot{\theta}_0 + \ddot{\theta}_1) + \epsilon^2((2\omega_2 + \omega_1^2)\ddot{\theta}_0 + 2\omega_1\ddot{\theta}_1 + \ddot{\theta}_2) = -\theta_0 - \epsilon\theta_1 - \epsilon^2\left(\theta_2 - \frac{1}{6}\theta_0^3\right) - \cdots$$

The initial conditions are as before. We find

$$\begin{aligned}
\mathcal{O}(1):&\quad \ddot{\theta}_0 = -\theta_0; \quad \theta_0(0) = 1, \quad \dot{\theta}_0(0) = 0\\
\mathcal{O}(\epsilon):&\quad 2\omega_1\ddot{\theta}_0 + \ddot{\theta}_1 = -\theta_1; \quad \theta_1(0) = 0, \quad \dot{\theta}_1(0) = 0\\
\mathcal{O}(\epsilon^2):&\quad (2\omega_2 + \omega_1^2)\ddot{\theta}_0 + 2\omega_1\ddot{\theta}_1 + \ddot{\theta}_2 = -\theta_2 + \frac{1}{6}\theta_0^3; \quad \theta_2(0) = 0, \quad \dot{\theta}_2(0) = 0\\
&\quad\vdots
\end{aligned}$$

By taking $\omega_1 = 0$, we find as before that

$$\theta_0(t) = \cos t, \quad \text{and} \quad \theta_1 = 0.$$

Note that if $\omega_1 \neq 0$, then

$$\ddot{\theta}_1 + \theta_1 = 2\omega_1 \cos t. \tag{5}$$

Since the right hand side solves the homogeneous equation, $\ddot{\theta} + \theta = 0$, any particular solution of (5) contains a non-zero term like

$$At\cos t + Bt\sin t.$$

That is an unwanted unbounded/secular term. Let us continue to determine θ_2:

$$\begin{aligned}
\ddot{\theta}_2 + \theta_2 &= \frac{1}{6}\theta_0^3 - (2\omega_2 + \omega_1^2)\ddot{\theta}_0 - 2\omega_1\ddot{\theta}_1\\
&= \frac{1}{6}\cos^3 t + 2\omega_2\cos t\\
&= \frac{1}{6}\frac{1}{4}(3\cos t + \cos 3t) + 2\omega_2\cos t\\
&= \frac{1}{24}\cos 3t + (\frac{3}{24} + 2\omega_2)\cos t \qquad (6)
\end{aligned}$$

Here again $\cos t$ solves the homogeneous equation, and unbounded/secular terms can only be avoided if $\omega_2 = -\frac{1}{2}\cdot\frac{3}{24} = -\frac{1}{16}$. In this case the particular solution has the form

$$\theta_2^p = C_1\cos 3t + C_2\sin 3t,$$

which solves (6) when $C_1 = -\frac{1}{192}$ and $C_2 = 0$. This general solution of (6) is then

$$\theta_2 = A\cos t + B\sin t - \frac{1}{192}\cos 3t$$

and initial conditions, $\theta_2(0) = 0 = \dot{\theta}_2(0)$ implies that

$$\theta_2 = \frac{1}{192}(\cos t - \cos 3t)$$

Hence we have found the following two approximations

$$\begin{aligned}\theta(t) &= \theta_0(t) + \mathcal{O}(\epsilon)\\ &= \cos(t) + \mathcal{O}(\epsilon)\end{aligned}$$

and

$$\begin{aligned}\theta(t) &= \theta_0\left(\left(1-\frac{\epsilon^2}{16}\right)t\right) + \epsilon^2\theta_2\left(\left(1-\frac{\epsilon^2}{16}\right)t\right) + \mathcal{O}(\epsilon^3)\\ &= \cos\left(\left(1-\frac{\epsilon^2}{16}\right)t\right) + \frac{\epsilon^2}{192}\left[\cos\left(\left(1-\frac{\epsilon^2}{16}\right)t\right) - \cos\left(3\left(1-\frac{\epsilon^2}{16}\right)t\right)\right] + \mathcal{O}(\epsilon^3)\end{aligned}$$

Note that there are no unbounded/secular terms anymore. These approximations can be expected to be good for all $t > 0$.

2 Inserting $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ into the equation and collecting terms of the same order of ε, we get

$$\ddot{x}_0 + \varepsilon\left(\ddot{x}_1 + 2\dot{x}_0 + x_0\right) + \varepsilon^2\left(\ddot{x}_2 + 2\dot{x}_1 + x_1\right) + \ldots = 0.$$

We hence get the following equations for x_0 and x_1:

$$\begin{aligned}\ddot{x}_0(t) &= 0,\\ \ddot{x}_1(t) &= -2\dot{x}_0(t) - x_0(t).\end{aligned}$$

For the initial conditions, it is natural to set

$$\begin{aligned}x_0(0) &= 0, \qquad \dot{x}_0(0) = 1,\\ x_1(0) &= 0, \qquad \dot{x}_1(0) = -1.\end{aligned}$$

Solving first for x_0, we get $x_0(t) = t$, which inserted into the second equation leads to

$$\ddot{x}_1(t) = -2 - t,$$

from which we get $x_1(t) = -\left(\frac{1}{6}t^3 + t^2 + 1\right)$. Hence,

$$x(t) = t - \varepsilon\left(\frac{1}{6}t^3 + t^2 + 1\right) + \mathcal{O}(\varepsilon^2).$$

3 We assume $y = y_0 + \varepsilon y_1 + \cdots$, which we insert into the equation, before collecting the terms of the same order of ε, to get

$$(\dot{y}_0 - y_0) + \varepsilon\left(\dot{y}_1 - y_1 - y_0^2 \mathrm{e}^{-t}\right) + \mathcal{O}(\varepsilon^2) = 0.$$

We hence get the following equations for y_0 and y_1:

$$\dot{y}_0(t) - y_0(t) = 0,$$
$$\dot{y}_1(t) - y_1(t) = y_0^2(t)\mathrm{e}^{-t}.$$

For the initial conditions, it is here natural to set $y_0(0) = 1$ and $y_1(0) = 0$.
Solving first for y_0, we get $y_0(t) = \mathrm{e}^t$, which inserted into the second equation leads to

$$\dot{y}_1(t) - y_1(t) = \mathrm{e}^t.$$

Multiplying both sides of the equation with e^{-t}, we get

$$\frac{d}{dt}\left(\mathrm{e}^{-t}y_1(t)\right) = 1.$$

so that

$$\mathrm{e}^{-t}y_1(t) = t + C.$$

From the initial condition we find $C = 0$, and thus

$$y_1(t) = t\mathrm{e}^t.$$

Collected, we get

$$y(t) = \mathrm{e}^t + \varepsilon t\mathrm{e}^t + \mathcal{O}(\varepsilon^2).$$

4 First we note that by the symmetry in the problem, u has to be an even function. To see this let $v(x) = u(-x)$ and observe that both v and u satisfies the equation. As u is even we have that $u'(-x) = -u'(x)$. Continuity of u' at 0 gives that $u'(0) = 0$ is the correct boundary condition.

By looking at the equation and boundary conditions we observe that for $x \approx 1$, $\epsilon u'' \approx -1$. Hence $|u''| \gg 1$ there, and if $u \sim \frac{1}{2}$ for x close to 0, u'' must be of order 1 close to $x = 0$. This means that we should use a boundary layer around $x = 1$. First we find the outer solution u_O. We neglect the term $\epsilon u''$ in the equation and obtain

$$(2 - x^2)u_O = 1, \text{ or } u_O(x) = \frac{1}{2 - x^2}.$$

We see that $u_O'(0) = 0$. Now we turn to the inner solution. Rescale $x = 1 - \delta\xi$ and $U(\xi) = u(x)$. This implies $\frac{1}{\delta^2}U''(\xi) = u''(x)$, which gives the equation

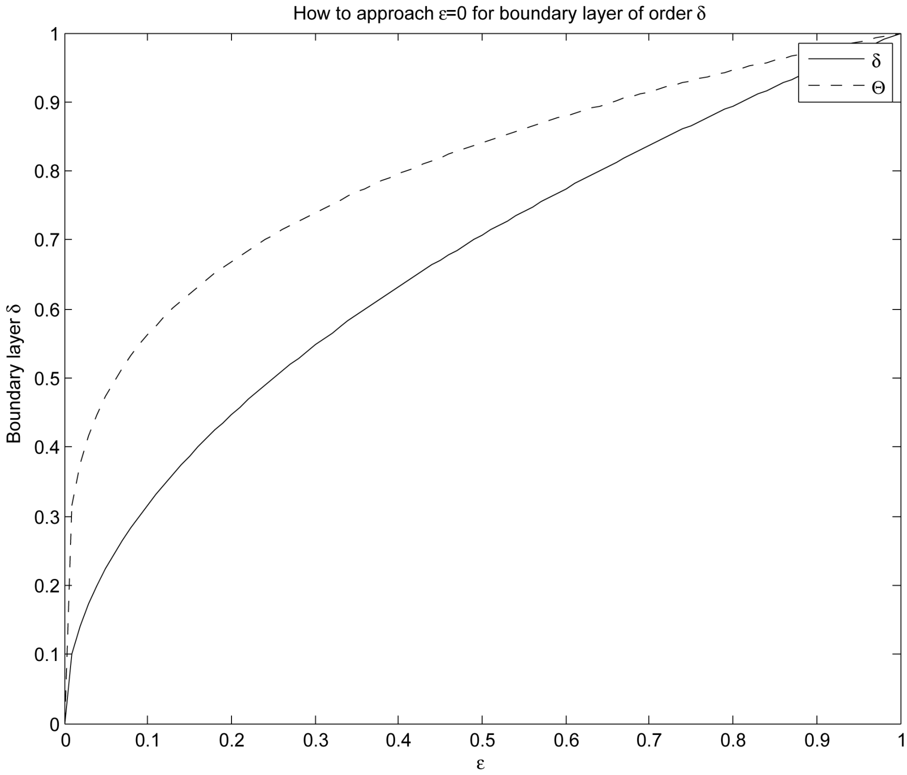
$$\frac{\epsilon}{\delta^2}U'' - \left(2 - (1 - \delta\xi)^2\right)U = -1$$
$$\frac{\epsilon}{\delta^2}U'' - \left(1 + \mathcal{O}(\delta)\right)U = -1.$$

We balance the terms by choosing $\delta = \epsilon^{\frac{1}{2}}$. The linear ODE has general solution $U(\xi) = c_1e^{\xi} + c_2e^{-\xi} + 1$. To determine c_1 and c_2 we need two boundary conditions. Observe first that $\xi = 0$ corresponds to $x = 1$. Thus $U(0) = u(1) = 0$, and the first equation is

$$c_1 + c_2 + 1 = 0.$$

We want the inner solution U and the outer solution u to match in a nice manner. That is, the transition from outside to inside the boundary layer should be continuous when $\epsilon \downarrow 0$. But as $\epsilon \downarrow 0$ the boundary layer shrinks as well. Let $\Theta(\epsilon)$ be a path

Figure 1: The path Θ in the plane.



in the plane such that $\Theta(\epsilon) \to 0$ as $\epsilon \downarrow 0$. The idea is to choose Θ such that for each $\epsilon > 0$ the function value $\Theta(\epsilon)$ lies in the intermediate region, at least for small ϵ. This happens if $\lim_{\epsilon \downarrow 0} \frac{\Theta(\epsilon)}{\delta(\epsilon)} = \infty$. See Figure 4. To sum up we have the following conditions on the function Θ

$$\lim_{\epsilon \downarrow 0} \Theta(\epsilon) = 0, \qquad \lim_{\epsilon \downarrow 0} \frac{\Theta(\epsilon)}{\delta(\epsilon)} = \infty.$$

The nice behaviour across the boundary layer can be formulated

$$\lim_{\epsilon \downarrow 0} u_O\left(1 - \eta\Theta(\epsilon)\right) = \lim_{\epsilon \downarrow 0} U\left(\frac{\eta\Theta(\epsilon)}{\delta(\epsilon)}\right),$$

for any parameter $\eta > 0$. In our case the left hand side will be equal to one. Thus

$$\lim_{\epsilon \downarrow 0} c_1 e^{\frac{\eta\Theta}{\sqrt{\epsilon}}} + c_2 e^{-\frac{\eta\Theta}{\sqrt{\epsilon}}} = 1 - 1 = 0.$$

This gives $c_1 = 0$ (or the exponential would go to infinity due to the limit of $\frac{\Theta}{\delta}$) and we get $c_2 = -1$. The uniform approximation is then given by

$$u_u(x) = \frac{1}{2 - x^2} + 1 - e^{-\frac{1-x}{\sqrt{\epsilon}}} - \lim_{\epsilon \downarrow 0} u_O(1 - \eta\Theta) = \frac{1}{2 - x^2} - e^{-\frac{1-x}{\sqrt{\epsilon}}}.$$