TMA4195
Mathematical Modelling
Autumn 2017

Exercise set 4
Exercise session 2017-9-21

1 (Exercise 2a p. 298 in Lin & Segel)
Find leading order outer, inner and uniform solutions to the following problem. (Assume that ε is small and positive and that the boundary layer is at $x = 0$.)

$$\begin{aligned} \varepsilon y'' + (1+x)\,y' + y &= 0, \\ y(0) &= 0, \\ y(1) &= 1. \end{aligned}$$

2 (Exercise 2b p. 298 in Lin & Segel)
Find leading order outer, inner and uniform solutions to the following problem. (Assume that ϵ is small and positive and that the boundary layer is at $x = 0$.)

$$\begin{aligned} \epsilon y'' + y' + y^2 &= 0, \\ y(0) &= \frac{1}{4}, \\ y(1) &= \frac{1}{2}. \end{aligned}$$

3 (Exercise 9.2.9 p.21 in Krogstad)
The cell density, n^*, in a part of the body can be modelled by

$$\frac{dn^*}{dt^*} = \alpha n^* - \omega n^*,$$

where α is the birth rate and ω the death rate. In order that cell growth should not come out of control, the cells produce an inhibitor that inhibits the uncontrolled growth. The inhibitor has density c^* and acts by regulating the birth rate:

$$\alpha = \frac{\alpha_0}{1 + c^*/A}.$$

The production of inhibitor is proportional to n^*, while it is degraded at a rate δ:

$$\frac{dc^*}{dt^*} = \beta n^* - \delta c^*.$$

This system has a time scale ω^{-1} related to the deaths of cells, and a time scale δ^{-1} related to related to chemical degradation of the inhibitor. We assume that $\omega^{-1} \gg \delta^{-1}$.

(a) Scale the system of equations using ω^{-1} as time and A as scale for c^*. Show that the scaled system for n^* is

$$\begin{aligned} \dot{n} &= \left(\frac{\kappa}{1+c} - 1\right) n, \\ \varepsilon \dot{c} &= n - c. \end{aligned} \tag{1}$$

What does ε and κ mean? What can we say about the size of ε, and what is a system of equations like this called? Decide the type of the trivial equilibrium point $(0,0)$. (In this part and the rest of the exercise we assume that κ is larger than 1).

(b) Find the path of the outer solution in the phase plane (i.e the curve described by $(n_O(t), c_O(t)))$ and the equation for $n_O(t)$. Show, without solving this equation, that all movement on this path is towards an equilibrium point for the full problem.

(c) Determine the leading order inner solution of (1) after rescaling in time, $\tau = \frac{t}{\varepsilon}$. Find a leading order uniform approximate solution.
(It is not possible to solve the equation in (b) explicitly).

4 In a model of a chemical tube reactor, the concentration c of the reactant satisfies the following reaction-diffusion equation:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + c(1-c), \quad t > 0,\ x \in \mathbb{R}. \tag{2}$$

a) Show that the linearization about $c = 0$ of equation (2) is given by

$$\frac{\partial c_L}{\partial t} = \frac{\partial^2 c_L}{\partial x^2} + kc_L, \quad t > 0,\ x \in \mathbb{R}. \tag{3}$$

Determine k.

b) Assume

$$c_L(x,0) = c_0(x), \quad x \in \mathbb{R}. \tag{4}$$

Show that the solution of (3)-(4) for any k is given by

$$\begin{aligned} c_L(x,t) &= e^{kt}(c_0 * c_F)(x,t) \\ &= e^{kt}\int_{-\infty}^{\infty} c_0(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy. \end{aligned}$$

Hint: Use that $c_F(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ satisfies

$$\frac{\partial c_F}{\partial t} = \frac{\partial^2 c_F}{\partial x^2}, \quad t > 0,\ x \in \mathbb{R}.$$

c) Show that

$$|c_L(x,t) - 0| \le e^{kt}\max_{x\in\mathbb{R}}|c_0(x) - 0|.$$

Hint: Use that $\int_{-\infty}^{\infty} c_F(x,t)dx = 1$.

d) Find all (constant) equilibrium points c_E of the equation (2). Determine whether they are stable or not according to linear stability analysis.
Hint: Consider only bounded (initial) perturbations $c_E + \tilde{c}_0(x)$, where

$$\max_{x\in\mathbb{R}}|\tilde{c}_0(x)| < \infty.$$

Remark: A dynamical system given by an ODE is a system where points in space move in time. A dynamical system given by a PDE is a system where points in a *function space* move in time. To discuss stability, we need to measure the distance between such points, e.g. by fixing a norm. For $\mathbb{R}^n$ and ODEs, this is not a problem, since all norms are equivalent in this case. For functions spaces, however, different norms are not equivalent, and we get *different* concepts of stability depending on our choice of norm. Here we choose the L^∞-norm, the simplest choice for our problem.