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Solutions to exercise set 4

1 We are going to study the boundary problem

$$(1) \qquad \varepsilon y'' + (1+x)y' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1$$

where $0 < \varepsilon \ll 1$. Let us introduce the notation $y = y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)$ for the outer approximation and $Y = Y_0 + \varepsilon Y_1 + \mathcal{O}(\varepsilon^2)$ for the inner approximation. From the exercise we know that the boundary layer lies at $x = 0$. Let us call the uniform approximation y^u.

To find the outer approximation, we plug in the series expansion of y into (1) and find

$$(1+x)y_0' + y_0 + \varepsilon[y_0'' + (1+x)y_1' + y_1] = \mathcal{O}(\varepsilon^2).$$

We can fix y_0 using the equation $y_0'(1+x) + y_0 = 0$. This equation has general solution of the form $y_0 = C/(1+x)$. If we use the initial condition $y(0) = 0$ for fixing C, we get $y_0 \equiv 0$. With further computations we will see that this solution can not be used as the inner solution. Since the boundary layer lies at $x = 0$, we are going to use the initial condition at $x = 1$ for the outer approximation. Thus, asking that $y_0(1) = 1$, we get $C = 2$. The leading order for the outer approximation is

$$y(x) \approx y_0(x) = \frac{2}{1+x}.$$

To find the inner approximation Y, we scale x such that the new variable varies between 0 and 1 close to the boundary layer. Let us set $\xi = x/\varepsilon$ and obtain, using (1),

$$(2) \qquad \frac{d^2Y}{d\xi^2} + \frac{dY}{d\xi} + \varepsilon\left(\xi\frac{dY}{d\xi} + Y\right) = 0.$$

Let us plug in the series expansion for Y into (2) and gather together the terms having same order. We get the following equation

$$\ddot{Y}_0 + \dot{Y}_0 + \varepsilon(\ddot{Y}_1 + Y_1 + \xi\dot{Y}_0 + Y_0) = \mathcal{O}(\varepsilon^2),$$

where $\dot{Y}$ means derivation with respect to ξ. We assume that this approximation is good near $\xi = 0$. We use the initial condition $Y(0) = 0$, that gives us $Y_0(0) = 0$. The initial term in the expansion of Y can be derived from the equation $\ddot{Y}_0 + \dot{Y}_0 = 0$. Using the initial condition $Y_0(0) = 0$, we get $Y_0 = K(1 - \mathrm{e}^{-\xi})$.

We have to match the inner and outer approximation. This can be done for Y_0 and y_0 by setting

$$\lim_{\xi\to\infty} Y(\xi) = \lim_{x\to 0} y(x),$$

which gives $K = 2$. The uniform approximation can be found summing together inner and outer and subtracting from their sum the common terms. This returns

$$y^u(x) = y(x) + Y(x/\varepsilon) - 2 = \frac{2}{1+x} - 2\mathrm{e}^{-x/\varepsilon}.$$

If we had used the initial condition $y(0) = 0$ instead of $y(1) = 1$, we would have found $y = 0$ and $K = 0 \Rightarrow Y = 0$. The uniform approximation would have been $y^u = 0$, which is clearly wrong. It is therefore right to use the initial condition $y(1) = 1$.

2 We begin by finding the outer solution. The boundary layer is given to be at $x = 0$. We neglect the $\epsilon y''$ term and obtain the boundary value problem for the outer solution

$$y_O' + y_O^2 = 0, \qquad y(1) = \frac{1}{2}.$$

The solution is $y_O(x) = \frac{1}{x+1}$.

To find the inner solution we scale $x = \delta\xi$ and let $Y(\xi) = y(x)$. The chain rule gives then

$$\frac{\epsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y^2 = 0, \qquad Y(0) = \frac{1}{4}.$$

We choose $\delta = \epsilon$ and multiply the equation by ϵ. This gives

$$Y'' + Y' + \epsilon Y^2 = 0.$$

The leading order solution can be found by solving $Y_I'' + Y_I' = 0$, $Y_I(0) = \frac{1}{4}$. The solution is $Y_I(\xi) = C_1 + C_2 e^{-\xi}$ with $C_1 + C_2 = \frac{1}{4}$. We use the matching requirement to find C_1 and C_2. To that end let $\Theta(\epsilon)$ be such that

$$\lim_{\epsilon\downarrow 0}\Theta(\epsilon) = 0, \qquad \lim_{\epsilon\downarrow 0}\frac{\Theta(\epsilon)}{\delta(\epsilon)} = \infty.$$

The matching requirement can be formulated

$$\lim_{\epsilon\downarrow 0} y_O\left(\eta\Theta(\epsilon)\right) = \lim_{\epsilon\downarrow 0} Y_I\left(\frac{\eta\Theta(\epsilon)}{\delta(\epsilon)}\right),$$

for all $\eta > 0$. This gives

$$C_1 + C_2 e^{-\infty} = 1,$$

and thus $C_1 = 1, C_2 = -\frac{3}{4}$. The uniform approximation is then given by

$$\begin{aligned} y_u(x) &= y_O(x) + Y_I\left(\frac{x}{\delta}\right) - \lim_{\epsilon\downarrow 0} y_O\left(\eta\Theta(\epsilon)\right) \\ &= \frac{1}{x+1} - \frac{3}{4}e^{-\frac{x}{\epsilon}}. \end{aligned}$$

3 (a) We use the time scale $T = 1/\omega$, and A as scale for c^*. By comparing the equations

$$\frac{dn^*}{dt^*} = \alpha n^* - \omega n^*,$$

and

$$\dot{n} = \left(\frac{\kappa}{1+c} - 1\right) n,$$
$$\varepsilon \dot{c} = n - c., \tag{3}$$

we see that we have to choose the scale for n^* to be

$$N_0 = \frac{\delta}{\beta} A, \tag{4}$$

and by using this we get equation 3. The parameters are

$$\kappa = \frac{\alpha_0}{\omega} = \frac{1/\omega}{1/\alpha_0},$$
$$\varepsilon = \frac{\omega}{\delta} = \frac{1/\delta}{1/\omega}. \tag{5}$$

Both are ratios between time scales, and it is given that κ is larger than 1, while $0 < \varepsilon \ll 1$. Hence, this is a singular perturbed system. The equilibrium points are:

$$(n_1, c_1) = (0, 0),$$
$$(n_2, c_2) = (\kappa - 1, \kappa - 1) \tag{6}$$

Linearisation around $(0, 0)$ gives

$$\begin{bmatrix} \kappa - 1 & 0 \\ \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{bmatrix}. \tag{7}$$

With eigenvalues $\lambda_1 = \kappa - 1 > 0$ and $\lambda_2 = -1/\varepsilon < 0$, is it a saddle point. It was not asked for to analyze the other equilibrium point.

(b) Equation 3 is a singular perturbed system because $\varepsilon \ll 1$. For the outer solution to leading order, $n_0(t)$ and $c_0(t)$, we find first that $n_0(t) = c_0(t)$, and this gives

$$\frac{dn_0}{dt} = \left(\frac{\kappa}{1+n_0} - 1\right) n_0. \tag{8}$$

The point $(\kappa - 1, \kappa - 1)$ is still an equilibrium, and by a sign analysis we see that $(\kappa - 1)$ is a stable equilibrium for (8) independent on where we choose to start for $0 < n_0(0) < \infty$. It is also possible to show that the point is locally stable by differentiate the right side of the equation.
It is also possible to partly solve 8 because we can write it as

$$\frac{1+n_0}{(\kappa - 1 - n_0)\, n_0} dn_0 = dt, \tag{9}$$

or if $\kappa > 1$

$$\left(\frac{1}{n_0} - \frac{\kappa}{n_0 - \kappa + 1}\right) dn_0 = \frac{dt}{\kappa - 1}. \tag{10}$$

Thus the solution is implicitly given by

$$\frac{n_0}{|n_0 - (\kappa - 1)|^{\kappa}} = e^{(t-t_0)/(\kappa-1)}. \tag{11}$$

Because $e^t \to \infty$ for $t \to \infty$, we always have $n_0(t) \underset{t\to\infty}{\longrightarrow} \kappa - 1$ (if $n_0(0) \neq 0$).
The outer path of the solution to leading order is the straight line $\{(n_0, c_0)\,; n_0 = c_0\}$.

(c) We try to use the time-scale ε, i.e. $\tau = t/\varepsilon$ for the beginning of the movement. This gives us the following equations, where we use $N(\tau)$ and $C(\tau)$ to separate the inner solution from the others:

$$\frac{dN}{d\tau} = \varepsilon\left(\frac{\kappa}{1+C} - 1\right)N,$$
$$\frac{dC}{d\tau} = N - C. \tag{12}$$

To leading order we get

$$\frac{dN_0}{d\tau} = 0,$$
$$\frac{dC_0}{d\tau} = N_0 - C_0,$$

We start in the point $(n(0), c(0))$, thus the inner solution

$$N_0(\tau) = n(0),$$
$$C_0(\tau) = [c(0) - n(0)]\, e^{-\tau} + n(0). \tag{13}$$

For the outer solution we don't have a starting point, but we know from (b) that $n_0(t) = c_0(t)$, hence we may assume

$$n_0(0) = A,$$
$$c_0(0) = A. \tag{14}$$

"Matching requirement" is in the simplest from

$$\lim_{\tau\to\infty} C_0(\tau) = \lim_{t\to 0} c_0(t),$$
$$\lim_{\tau\to\infty} N_0(\tau) = \lim_{t\to 0} n_0(t), \tag{15}$$

and luckily $A = n(0)$ satisfies both of these requirements. Uniform solution is generally given as

$$u_u(t) = u_0(t) + U_0(\tau) - \lim_{\tau\to\infty} U_0(\tau). \tag{16}$$

Thus we have

$$n_u(t) = n_0(t),$$
$$c_u(t) = n_0(t) + [c(0) - n(0)]\, e^{-t/\varepsilon}. \tag{17}$$

Generally

$$\dot{c} = \frac{1}{\varepsilon}(n - c) \tag{18}$$

gives that $\dot{c}$ is large and positive if $n - c \gg \mathcal{O}(\varepsilon)$, and large and negative if $n - c \ll \mathcal{O}(-\varepsilon)$. Further we see that $\dot{n} < 0$ if $c > \kappa - 1$, and $\dot{n} > 0$ if $c < \kappa - 1$. Together with the results from (b), this gives a good quantitative impression of the paths, which in Figure 1 is computed numerically. It is obvious that as long as we don't start with $n(0) = 0$ we will for $t \to \infty$ end up in the equilibrium point $(\kappa - 1, \kappa - 1)$.

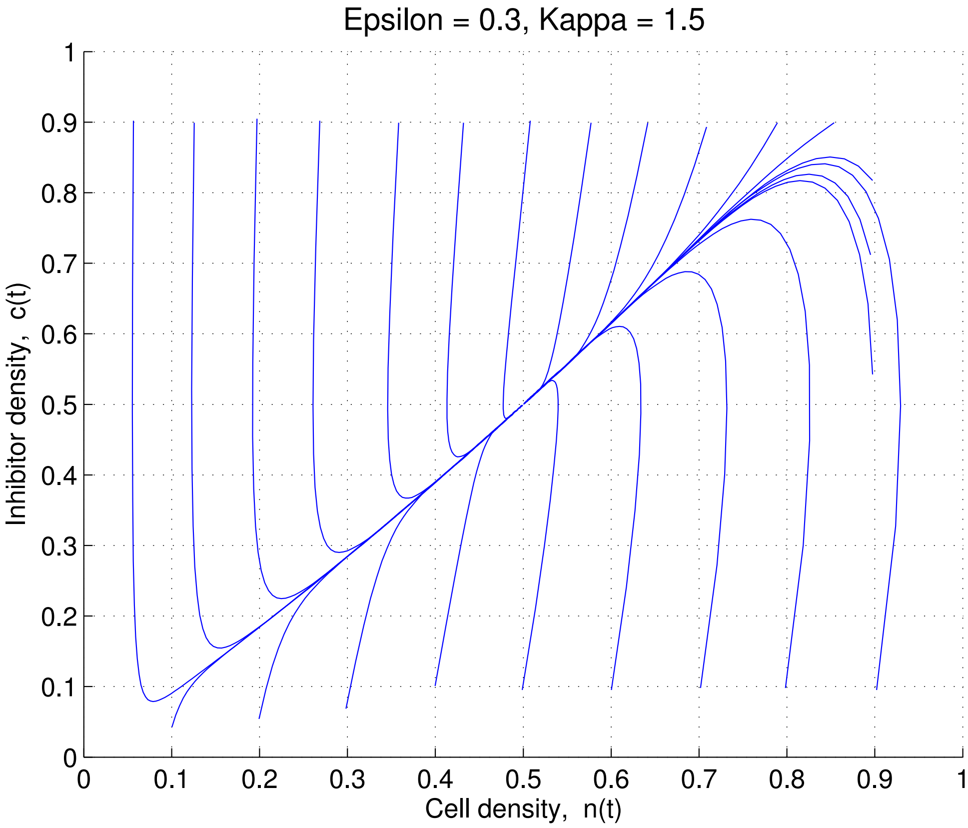

Figure 1: Paths for the system when $\varepsilon = 0.3$ and $\kappa = 1.5$. The stable equilibrium point is in $(\kappa - 1, \kappa - 1)$.

4 We are given the equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + c(1-c), \quad t > 0,\ x \in \mathbb{R}. \tag{19}$$

a) We linearize the equation around $c = 0$. It is more transparent to write the equation as

$$c_t = c_{xx} + q(c).$$

The only term we need to linearize is q, since the other terms are already linear:

$$q(c) \approx q(0) + q'(0)c = 0 + 1 \cdot c = c,$$

and hence the linearized equation is

$$c_t = c_{xx} + c, \tag{20}$$

where we have set $c := c_L(x,t)$. We thus have $k = 1$.

b) We can solve the linearized equation in many ways, but following the hint, we want to transform it to the heat equation. We do this using an integrating factor. Let $\bar{c} = e^{-kt}c$ and note that

$$\frac{\partial}{\partial t}\bar{c} = e^{-kt}(c_t - kc) = e^{-kt}c_{xx} = \bar{c}_{xx}.$$

This can be solved by convolution with the fundamental solution c_F:

$$\bar{c} = \bar{c}_0 * c_F = \int_{-\infty}^{\infty} \bar{c}(y,0)c_F(x-y,t)\mathrm{d}y.$$

We then get

$$c_L(x,t) = \mathrm{e}^{kt}\bar{c}(x,t) = \mathrm{e}^{kt}\int_{-\infty}^{\infty} c_0(y)c_F(x-y,t)\mathrm{d}y,$$

where we note that $\bar{c}(y,0) = e^{-0}c(y,0) = c_0(y)$.
Inserting the given solution of the fundamental solution c_F, we get

$$c_L(x,t) = e^{kt}\int_{-\infty}^{\infty} c_0(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy.$$

c) Using the hint, we calculate

$$|c_L - 0| \leq \mathrm{e}^{kt}\int_{-\infty}^{\infty}|c_0(y)|c_F(x-y,t)\mathrm{d}y \leq \mathrm{e}^{kt}\max_{x\in\mathrm{R}}|c_0(x)| \cdot 1 = \mathrm{e}^{kt}\max_{x\in\mathrm{R}}|c_0(x) - 0|.$$

d) The equilibrium points of (19) are its constant solutions, and if $c = c_E$ is a constant solution of (19), then $(c_E)_t = (c_E)_{xx} = 0$ and $q(c_E) = c_E(c_E - 1) = 0$. The solutions/equilibrium points are therefore $c_E = 0$ and $c_E = 1$.
To study the stability of the equilibrium points c_E, we check whether solutions of the equation linearized arond c_E that start near c_E remain near for all times. To do that, let

$$c(x,t) = c_E + \tilde{c}(x,t)$$

and note that if $\tilde{c}$ is not so big, then

$$\tilde{c}_t = \tilde{c}_{xx} + q(c_E + \tilde{c}) \approx \tilde{c}_{xx} + q(c_E) + q'(c_E)\tilde{c}.$$

Note that $q(c_E) = 0$ and let $\hat{c}$ be the solution of the linearized equation

$$c_t = c_{xx} + q'(c_E)c. \tag{21}$$

This linearized equation only has the equilibrium point $\hat{c} = 0$ (since $q' \neq 0$). By definition we say that c_E is a stable(/unstable) equilibrium point of the original non-linear equation according to linear stability analysis if $\hat{c} = 0$ is a stable(/unstable) equilibrium point of the linearized equation 21.
We solve equation (21) and $c(x, 0) = c_0(x)$ as in part b), this time with using the integrating factor $e^{-q'(c_e)t}$:

$$\hat{c}(x, t) = e^{q'(c_e)t} \int_{\mathbb{R}} c_0(y) c_F(x - y, t) dy.$$

Note that if $|c_0(x) - 0| = |c_0| < \delta$, then

$$|\hat{c}(x, t) - 0| \leq \max_{x \in \mathrm{R}} |c_0(x) - 0| < \delta e^{q'(c_e)t}.$$

Hence it follows that $\hat{c} = 0$ is a stable equilibrium point if $q'(c_e) \leq 0$ since then small perturbations remain small for all times. On the other hand, if $q'(c_e) > 0$, then $\hat{c} = 0$ is not stable any more since we can find small perturbations that blows up in time. Take e.g. $c_0 = \delta$ and check that

$$\hat{c}(x, t) = \delta e^{q'(c_e)t} \to \infty \qquad \text{as} \qquad t \to \infty.$$

We compute q' and find that $q'(0) = 1 > 0$ and $q'(1) = -1 < 0$. From the discussion above we can then conclude according to linear stability analysis that $c_E = 0$ is unstable while $c_E = 1$ is stable.