Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4195
Mathematical Modelling
Autumn 2017

Exercise set 5
Exercise session 2017-9-28

$\boxed{1}$ (Exercise 2.2 p. 377 in Logan) Determine the equilibrium solutions and sketch a bifurcation diagram of the following differential equations. Identify the bifurcation points and bifurcation solutions. Investigate the stability of the equilibrium solutions and indicate where an exchange of stability occurs.

(a)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (u-\mu)(u^2-\mu)$$

(b)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u(9-\mu u)(\mu+2u-u^2)$$

$\boxed{2}$ (Exercise 9.1.4 p. 18 in Krogstad) The simplest model for the earth's mean temperature may be written

$$C\frac{dT}{dt} = Q_0 - \sigma T^4$$

(C and σ are constants). The left side of the equation is the change with time in the stored heat, Q_0 is the mean heat radiation from the sun revceived by the earth, whereas the last term is heat loss by radiation into space (the Stefan-Boltzmann law of black body radiation). All quantities are expressed per unit surface area. In this problem we shall use $Q_0/\sigma = (287\text{K})^4$ and $C/\sigma = 6\times 10^9 \text{days}\times\text{K}^3$.

a) Find the stationary temperature, show that it is stable, and find (an estimate of) the time it takes for some deviation from the equilibrium temperature to die out. (OBS!! The answer depends on your definition of "die out".)

The amount of absorbed radiation from the sun depends of, among others things, on the color of the earth's surface (*albedo*). During ice ages, when a relative larger part of the earth is covered with ice and snow, the mean absorbed energy will be less. The term Q_0 is therefore suggested to be modified to

$$Q_0 + Q_a \tanh\left(\frac{T-T_0}{T_n}\right),$$

where $Q_0 - Q_a$ represents ice ages, $Q_0 + Q_a$ warm periods, and T_0 the equilibrium temperature from (a). The temperature T_n controls the width of the transition range.

b) Discuss the modified model, and explain in particular what can happen if

$$T_n < \frac{Q_a}{4Q_0}T_0$$

and Q_0 varies.
Hint: Make a sketch and recall that

$$\tanh(x) \underset{x\to-\infty}{\longrightarrow} -1,$$
$$\tanh(x) = x + O\left(x^3\right),$$
$$\tanh(x) \underset{x\to\infty}{\longrightarrow} 1.$$

3 In a model of a chemical tube reactor, the concentration c of the reactant satisfies the following reaction-diffusion equation:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + c(1-c), \quad t>0,\ x\in\mathbb{R}. \tag{1}$$

a) Show that the linearization about $c=0$ of equation (1) is given by

$$\frac{\partial c_L}{\partial t} = \frac{\partial^2 c_L}{\partial x^2} + kc_L, \quad t>0,\ x\in\mathbb{R}. \tag{2}$$

Determine k.

b) Assume

$$c_L(x,0) = c_0(x), \quad x\in\mathbb{R}. \tag{3}$$

Show that the solution of (2)-(3) for any k is given by

$$\begin{aligned} c_L(x,t) &= e^{kt}(c_0 * c_F)(x,t) \\ &= e^{kt}\int_{-\infty}^{\infty} c_0(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy. \end{aligned}$$

Hint: Use that $c_F(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ satisfies

$$\frac{\partial c_F}{\partial t} = \frac{\partial^2 c_F}{\partial x^2}, \quad t>0,\ x\in\mathbb{R}.$$

c) Show that

$$|c_L(x,t)-0| \le e^{kt}\max_{x\in\mathbb{R}}|c_0(x)-0|.$$

Hint: Use that $\int_{-\infty}^{\infty} c_F(x,t)dx = 1$.

d) Find all (constant) equilibrium points c_E of the equation (1). Determine whether they are stable or not according to linear stability analysis.
Hint: Consider only bounded (initial) perturbations $c_E + \tilde{c}_0(x)$, where

$$\max_{x\in\mathbb{R}}|\tilde{c}_0(x)| < \infty.$$

Remark: A dynamical system given by an ODE is a system where points in space move in time. A dynamical system given by a PDE is a system where points in a *function space* move in time. To discuss stability, we need to measure the distance between such points, e.g. by fixing a norm. For $\mathbb{R}^n$ and ODEs, this is not a problem, since all norms are equivalent in this case. For functions spaces, however, different norms are not equivalent, and we get *different* concepts of stability depending on our choice of norm. Here we choose the L^∞-norm, the simplest choice for our problem.