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TMA4195
Mathematical Modelling
Autumn 2017

Solutions to exercise set 5

$\boxed{1}$ (a) The equilibrium points are given by

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad \begin{cases} u_0 = \mu \\ u_1 = \sqrt{\mu}, \quad \mu \geq 0 \\ u_2 = -\sqrt{\mu}, \quad \mu \geq 0 \end{cases}$$

We have two bifurcation points, i.e. points where the solution curves intersect. These points are given by $\mu = \pm\sqrt{\mu}$, that is the two bifurcation points are $(0,0), (1,1)$. To say something about stability, we first calculate

$$f'(u) = 3u^2 - 2\mu u - \mu$$

By plugging in the expression for the equilibrium solutions we get

$$\begin{aligned} f'(u_0) &= \mu(\mu - 1) \\ f'(u_1) &= 2\mu(1 - \sqrt{\mu}) \\ f'(u_2) &= 2\mu(1 + \sqrt{\mu}) \end{aligned}$$

We see that u_3 is always unstable, u_2 is stable for $\mu > 1$ and u_1 is stable when $0 < \mu < 1$. We see that the stability changes at the bifurcation points. See Figure 1 for the sketch of the bifurcation diagram.

(b) The equilibrium points are given by

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad \begin{cases} u_0 = 0 \\ u_1 = 9/\mu, \quad \mu \neq 0 \\ u_{2,3} = 1 \pm \sqrt{\mu + 1}, \quad \mu \geq -1 \end{cases}$$

The branching diagram is showed in figure 2.

From the diagram we see that we have two bifurcation points, i.e. points where the solution curves intersect. These points are given by

$$\begin{aligned} 1 - \sqrt{\mu + 1} &= 0 \\ 1 + \sqrt{\mu + 1} &= 9/\mu \end{aligned}$$

Hence the bifurcation points are $(\mu, u) = \{(0,0), (3,3)\}$.

To say something about stability, we first calculate

$$f'(u) = (9 - \mu u)(\mu + 2u - u^2) - \mu u(\mu + 2u - u^2) + 2u(9 - \mu u)(1 - u)$$

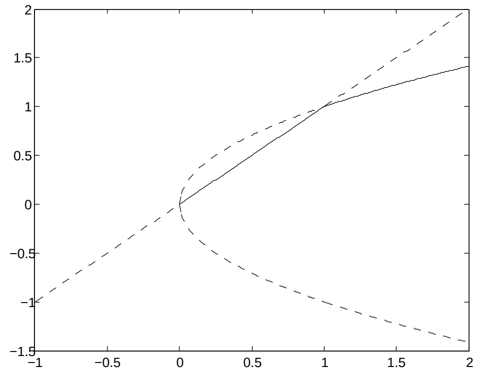

Figure 1: Bifurcation diagram 2.2 (a).

By plugging in the expression for the equilibrium solutions we get

$$\begin{aligned}
f'(u_0) &= 9\mu \\
f'(u_1) &= -9\left(\mu + \frac{18}{\mu} - \frac{81}{\mu^2}\right) \\
&= -\frac{9}{\mu^2}\left(\mu^3 + 18\mu - 81\right) \\
&= -\frac{9}{\mu^2}(\mu - 3)(\mu^2 + 3\mu + 27) \\
f'(u_2) &= 2\left(1 - \sqrt{\mu + 1}\right)\left[9 - \mu\left(1 - \sqrt{\mu + 1}\right)\right]\sqrt{\mu + 1} \\
f'(u_3) &= -2\left(1 + \sqrt{\mu + 1}\right)\left[9 - \mu\left(1 + \sqrt{\mu + 1}\right)\right]\sqrt{\mu + 1}.
\end{aligned}$$

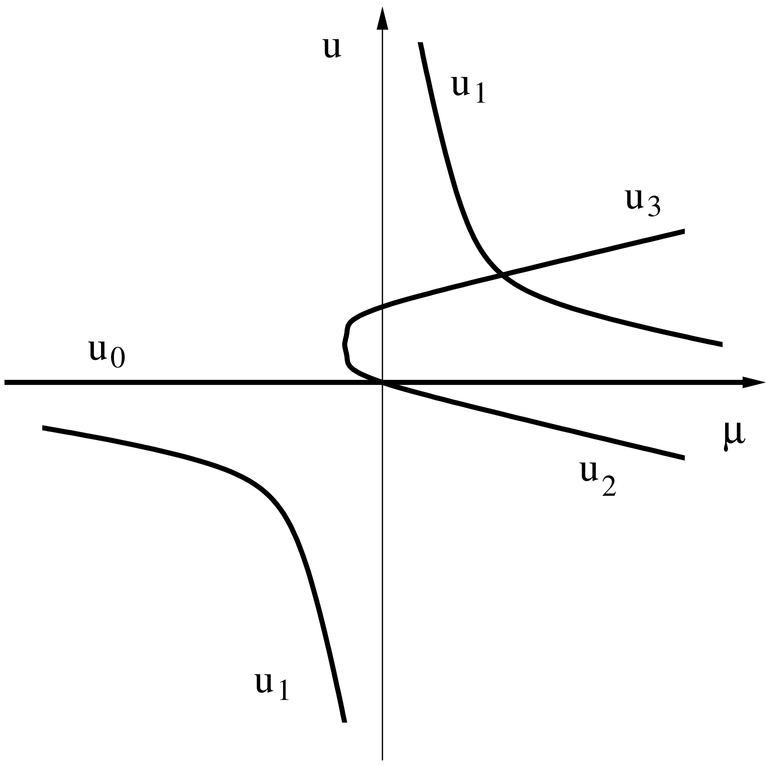

Figure 2: Branching diagram 2.2 (b).

By sketching a sign line for the expression on the right hand side we find that

$$u_0 \text{ is } \begin{cases} \text{stable for } \ \mu < 0 \\ \text{unstable for } \mu > 0, \end{cases}$$

$$u_1 \text{ is } \begin{cases} \text{stable for } \ \mu > 3 \\ \text{unstable for } \mu \in \langle -\infty, 0\rangle \cup \langle 0, 3\rangle, \end{cases}$$

$$u_2 \text{ is } \begin{cases} \text{stable for } \ \mu > 0 \\ \text{unstable for } \mu \in \langle -1, 0\rangle, \end{cases}$$

$$u_3 \text{ is } \begin{cases} \text{stable for } \ \mu \in \langle -1, 3\rangle \\ \text{unstable for } \mu > 3. \end{cases}$$

We see from this that the stability changes in the bifurcation points.

$\boxed{2}$

(a) We find the equilibrium solution using

$$Q_0 - \sigma T^4 = 0,$$

In other words

$$T_0 = \sqrt[4]{\frac{Q_0}{\sigma}} = 287\text{K}.$$

Let us check that T_0 is stable:

$$\frac{d}{dt}\left(\frac{Q_0 - \sigma T^4}{C}\right) = -\frac{1}{C}\sigma 4 T_0^3 < 0.$$

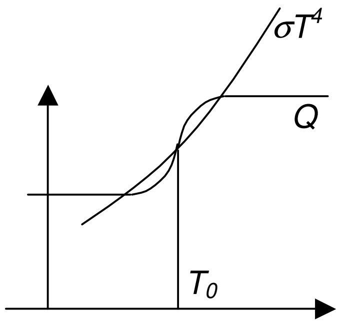

Figure 3: The Figure illustrates the equilibrium solutions when $T_n < \frac{Q_a}{4Q_0}T_0$. Only the solutions in the ends are stable.

Around T_0 we can linearize the equation and write $T = T_0 + y$. This gives us $\dot{y} = \left(-\frac{1}{C}\sigma 4T_0^3\right) y$. Using a perturbation like $y(0) = y_0$, the solution becomes

$$y(t) = y_0 \exp\left(-\frac{1}{C}\sigma 4T_0^3 t\right).$$

We solve for t and find that

$$t = -\frac{C}{\sigma 4T_0^3} \ln\left(\frac{y(t)}{y_0}\right).$$

If we say that the deviation has died out when $y(t)/y_0 \leq 10^{-5}$ (a relative estimate), then the time it needs to die out is

$$t = \frac{C}{\sigma 4T_0^3} 5 = 5\frac{6 \times 10^9 \text{days} \times \text{K}^3}{4\,(287\text{K})^3} \approx 317\text{days}.$$

(OBS!! The answer depends on your definition of "die out".)

(b) An equilibrium solution T_s solves the equation

$$f(T) = Q_0 + Q_a \tanh\left(\frac{T - T_0}{T_n}\right) - \sigma T^4 = 0,$$

and stability can be decided by checking that $f'(T_s)$ is bigger or smaller than zero. For the equilibrium temperature (a) (using the hint about tanh) we get

$$f'(T_0) = \frac{Q_a}{T_n} - 4\sigma T_0^3 = \frac{Q_a}{T_n} - \frac{4}{T_0}Q_0 = \frac{Q_a}{T_n}\left(1 - \frac{4Q_0 T_n}{T_0 Q_a}\right).$$

Whenever $T_n < \frac{Q_a}{4Q_0}T_0$, $f'(T_0) > 0$ and T_0 is not stable. The situation is as in fig. 3.

What controls the climate changes on earth are the variations in Q_0 that make the quantity $Q_0 + Q_a \tanh\left(\frac{T-T_0}{T_n}\right)$ bob up and down. Whenever $T_n < \frac{Q_a}{4Q_0}T_0$, is not surprising that transitions between "cold" and "warm" periods verify abruptly.

One can also think that $\sigma\varepsilon T^4$ varies whenever ε varies: that is, there is a connection between ε and the first term in the expansion (ε is the emissivity which measures the deviation from the ideal black body radiation corresponding to $\varepsilon = 1$).

Further investigations on this model, which was introduce in 1987 by Ghil and Childress, are left to the interested reader.

3 We are given the equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + c(1-c), \quad t > 0,\ x \in \mathbb{R}. \tag{1}$$

a) We linearize the equation around $c = 0$. It is more transparent to write the equation as

$$c_t = c_{xx} + q(c).$$

The only term we need to linearize is q, since the other terms are already linear:

$$q(c) \approx q(0) + q'(0)c = 0 + 1 \cdot c = c,$$

and hence the linearized equation is

$$c_t = c_{xx} + c, \tag{2}$$

where we have set $c := c_L(x,t)$. We thus have $k = 1$.

b) We can solve the linearized equation in many ways, but following the hint, we want to transform it to the heat equation. We do this using an integrating factor. Let $\bar{c} = e^{-kt}c$ and note that

$$\frac{\partial}{\partial t}\bar{c} = e^{-kt}(c_t - kc) = e^{-kt}c_{xx} = \bar{c}_{xx}.$$

This can be solved by convolution with the fundamental solution c_F:

$$\bar{c} = \bar{c}_0 * c_F = \int_{-\infty}^{\infty} \bar{c}(y,0)c_F(x-y,t)\mathrm{d}y.$$

We then get

$$c_L(x,t) = \mathrm{e}^{kt}\bar{c}(x,t) = \mathrm{e}^{kt}\int_{-\infty}^{\infty} c_0(y)c_F(x-y,t)\mathrm{d}y,$$

where we note that $\bar{c}(y,0) = e^{-0}c(y,0) = c_0(y)$.
Inserting the given solution of the fundamental solution c_F, we get

$$c_L(x,t) = e^{kt}\int_{-\infty}^{\infty} c_0(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy.$$

c) Using the hint, we calculate

$$|c_L - 0| \le \mathrm{e}^{kt}\int_{-\infty}^{\infty}|c_0(y)|c_F(x-y,t)\mathrm{d}y \le \mathrm{e}^{kt}\max_{x\in\mathrm{R}}|c_0(x)| \cdot 1 = \mathrm{e}^{kt}\max_{x\in\mathrm{R}}|c_0(x) - 0|.$$

d) The equilibrium points of (1) are its constant solutions, and if $c = c_E$ is a constant solution of (1), then $(c_E)_t = (c_E)_{xx} = 0$ and $q(c_E) = c_E(c_E - 1) = 0$. The solutions/equilibrium points are therefore $c_E = 0$ and $c_E = 1$.

To study the stability of the equilibrium points c_E, we check whether solutions of the equation linearized arond c_E that start near c_E remain near for all times. To do that, let

$$c(x, t) = c_E + \tilde{c}(x, t)$$

and note that if $\tilde{c}$ is not so big, then

$$\tilde{c}_t = \tilde{c}_{xx} + q(c_E + \tilde{c}) \approx \tilde{c}_{xx} + q(c_E) + q'(c_E)\tilde{c}.$$

Note that $q(c_E) = 0$ and let $\hat{c}$ be the solution of the linearized equation

$$c_t = c_{xx} + q'(c_E)c. \tag{3}$$

This linearized equation only has the equilibrium point $\hat{c} = 0$ (since $q' \neq 0$). By definition we say that c_E is a stable(/unstable) equilibrium point of the original non-linear equation according to linear stability analysis if $\hat{c} = 0$ is a stable(/unstable) equilibrium point of the linearized equation 3.

We solve equation (3) and $c(x, 0) = c_0(x)$ as in part b), this time with using the integrating factor $e^{-q'(c_e)t}$:

$$\hat{c}(x, t) = e^{q'(c_e)t} \int_{\mathbb{R}} c_0(y) c_F(x - y, t) dy.$$

Note that if $|c_0(x) - 0| = |c_0| < \delta$, then

$$|\hat{c}(x, t) - 0| \leq \max_{x \in \mathrm{R}} |c_0(x) - 0| < \delta e^{q'(c_e)t}.$$

Hence it follows that $\hat{c} = 0$ is a stable equilibrium point if $q'(c_e) \leq 0$ since then small perturbations remain small for all times. On the other hand, if $q'(c_e) > 0$, then $\hat{c} = 0$ is not stable any more since we can find small perturbations that blows up in time. Take e.g. $c_0 = \delta$ and check that

$$\hat{c}(x, t) = \delta e^{q'(c_e)t} \to \infty \qquad \text{as} \qquad t \to \infty.$$

We compute q' and find that $q'(0) = 1 > 0$ and $q'(1) = -1 < 0$. From the discussion above we can then conclude according to linear stability analysis that $c_E = 0$ is unstable while $c_E = 1$ is stable.